Hořava-Lifshitz Quantum Cosmology

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Arrabal - Jorge Luis Borges

(...)
los naipes de colores del poniente
Y sentí *Buenos Aires*.
Esta ciudad que yo creí mi pasado
es mi porvenir, mi presente;
los años que he vivido en Europa son ilusorios,
yo estaba siempre (y estaré) en Buenos Aires.

Jorge Luis Borges (1899–1986)

Fervor de Buenos Aires

in Obras Completas 1, Sudamericana (2011)

1 - Introduction

Hořava-Lifshitz (HL) Gravity

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- The Lorentz symmetry must be abandoned at high-energies.
- The novelty of this proposal is the breaking of Lorentz symmetry occurs the very way as in some condensed matter models:
  - Through the anisotropic scaling between space and time, namely $\vec{r} \to b\vec{r}$ and $t \to b^z t$, with b a scale parameter.
  - The dynamical critical exponent z is chosen in order to ensure that the gravitational coupling is dimensionless.
  - Less symmetries: foliation preserving diffeomorphism.

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  - The dynamical critical exponent z is chosen in order to ensure that the gravitational coupling is dimensionless.
  - Less symmetries: foliation preserving diffeomorphism.
- As GR is recovered at low-energies, Lorentz symmetry is recovered at the IR fixed point, z flows to $z = 1$ in this limit.
- Lots of problems...
  (Some of them also plague other quantum gravity models!)

Quantum Cosmology (QC)

- Quantum cosmology (QC) is an interesting step toward the understanding of quantum gravity and the initial conditions of the universe.
- It setup consists of splitting space-time using the Arnowitt-Deser-Misner (ADM) and applying the well-known quantum mechanical considerations for constrained systems.
- The cosmological principle is evoked so that the space-time is foliated in leaves with a constant global time.
- To implement the quantum scenario:
    - one promotes the Hamiltonian constraint $H = 0$ to an operator equation, the Wheeler-deWitt (WdW) equation:

$$\hat{H}\psi = 0, \text{ where } \psi \text{ is the wave function of the universe.}$$

HL Gravity in a Cosmological Context

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- One subtle point concerns the lapse function $N(t)$ implies that the classical Hamiltonian constraint of GR is no longer local.
- QC allows for a valuable insight of HL gravity in the quantum context.
  - In both approaches, one foliates the space-time in constant global time leaves.
  - One faces the problem of turning the Hamiltonian constraint into the WdW equation, since the Hamiltonian constraint is not local.
  - HL gravity does not introduce higher than first order time derivatives of the action.
  - HL gravity introduces only higher spatial derivative terms, which dominate on very small scales.

Aims

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- A matter section is not introduced, given that the main interest:
  - in the very early Universe, where the HL gravity dominate
  - the late Universe, an epoch dominated by the cosmological constant.

2 -The Wheeler-deWitt Equation

Metric

- One considers the Robert-Walker (RW) metric with $\mathbb{R} \times S^3$ topology

$$ds^2 = \sigma^2 \left(-N(t)^2 dt^2 + a^2 \gamma_{ij} dx^i dx^j\right) ,$$

  - where $i, j = 1, 2, 3$, σ^2 is a normalisation constant,
  - $N(t)$ is the lapse function
  - γ_{ij} is the metric of the unit 3-sphere. Its metric is given by $\gamma_{ij} = \text{diag}\left(\frac{1}{1-r^2}, r^2, r^2 \sin^2\theta\right)$.

Hořava-Lifshitz minisuperspace action

- General relativity action in a (3+1)-language

$$S_{\mathrm{HL}} = \frac{M_{\mathrm{Pl}}^2}{2} \int d^3x N \sqrt{g} \left\{ K_{ij} K^{ij} - K^2 + R - 2\Lambda \right\}$$

Hořava-Lifshitz minisuperspace action

- Introducing the *foliation-preserving diffeomorphism*

$$S_{\mathrm{HL}} = \frac{M_{\mathrm{Pl}}^2}{2} \int d^3x N \sqrt{g} \left\{ K_{ij} K^{ij} - \lambda K^2 + R - 2\Lambda \right\}$$

Hořava-Lifshitz minisuperspace action

- Adding *all* relevant couplings...

$$
\begin{aligned}
S_{\text{HL}} = \frac{M_{\text{Pl}}^2}{2} \int d^3x N \sqrt{g} \{ & K_{ij} K^{ij} - \lambda K^2 + R - 2\Lambda - g_2 M_{\text{Pl}}^{-2} R^2 \\
& - g_3 M_{\text{Pl}}^{-2} R_{ij} R^{ij} - g_4 M_{\text{Pl}}^{-4} R^3 - g_5 M_{\text{Pl}}^{-4} R \left(R^i{}_j R^j{}_i \right) \\
& - g_5 M_{\text{Pl}}^{-4} R \left(R^i{}_j R^j{}_i \right) - g_6 M_{\text{Pl}}^{-4} R^i{}_j R^j{}_k R^k{}_i - g_7 M_{\text{Pl}}^{-4} R \nabla^2 R \\
& - g_8 M_{\text{Pl}}^{-4} \nabla_i R_{jk} \nabla^i R^{jk} \} .
\end{aligned}
$$

- It is the action for the projectable HL gravity without detailed balance.

Hořava-Lifshitz minisuperspace action

- Substituting the RW metric into the HL action one finds the minisuperspace action:

$$S_{\mathrm{HL}} = \frac{1}{2}\int dt N \left\{ \frac{-\dot{a}^2 a}{N^2} + \frac{2a}{(3\lambda-1)} - \frac{\Lambda M_{\mathrm{Pl}} a^3}{18\pi^2(3\lambda-1)^2} - \frac{24\pi^2(3g_2+g_3)}{a} - \frac{288\pi^4(3\lambda-1)(9g_4+3g_5+g_6)}{a^3} \right\}.$$

- Redefining some coupling constants the HL minisuperspace action is finally written as

$$S_{\mathrm{HL}} = \frac{1}{2}\int \left(\frac{N}{a}\right)\left[-\left(\frac{a\dot{a}}{N}\right)^2 + g_C a^2 - g_\Lambda a^4 - g_r - \frac{g_s}{a^2}\right].$$

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Hořava-Lifshitz minisuperspace Hamiltonian and the Wheeler-DeWitt equation

- The HL minisuperspace Hamiltonian density is given by

$$H = \frac{1}{2}\frac{N}{a}\left(-\Pi_a^2 - g_C a^2 + g_\Lambda a^4 + g_r + \frac{g_s}{a^2}\right).$$

- The canonical quantisation is obtained by promoting the canonical conjugate momentum into an operator: $\Pi_a \mapsto -i\frac{d}{da}$.
- The Wheeler-deWitt (WdW) equation reads

$$\left\{\frac{d^2}{da^2} - g_C a^2 + g_\Lambda a^4 + g_r + \frac{g_s}{a^2}\right\}\Psi(a) = 0.$$

- This equation is similar to the one dimensional Schrödinger equation for $\hbar = 1$ and a particle with $m = 1/2$ with $E = 0$ and potential

$$V(a) = g_C a^2 - g_\Lambda a^4 - g_r - \frac{g_s}{a^2}.$$

Hořava-Lifshitz minisuperspace potentials

- Classically the allowed regions are such that $V(a) \leq 0$ since $E = 0$.
- The new terms introduced by HL gravity are the last two terms which dominate for $a \ll 1$, presumably at the early Universe where the GR description must be replaced by the quantum gravity one.
- At the very early Universe, this potential is dominated by the term $-g_s/a^2$, implying that $g_s < 0$.
- The potential exhibits a “barrier” that might prevent space-time from being singular.
- The choice g_s splits the discussion into three distinct scenarios:
  - $\Lambda = 0$
  - $\Lambda > 0$
  - $\Lambda < 0$

Hořava-Lifshitz minisuperspace potentials: $\Lambda = 0$

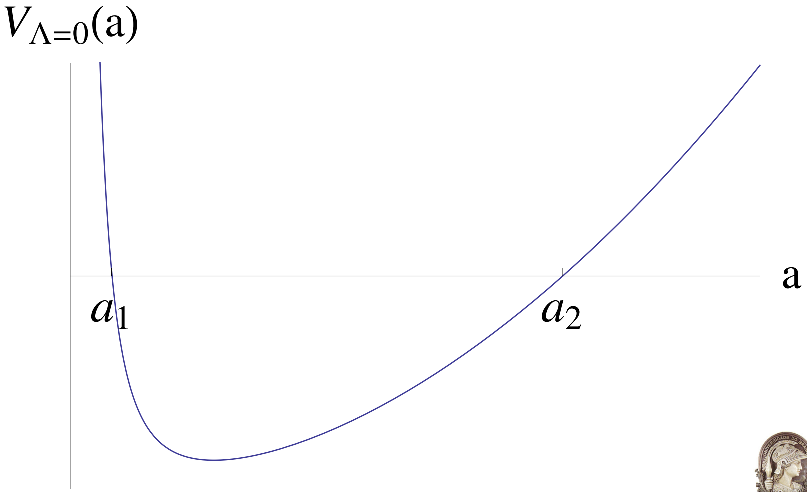


Figure: Potential for $\Lambda = 0$

Hořava-Lifshitz minisuperspace potentials: $\Lambda < 0$

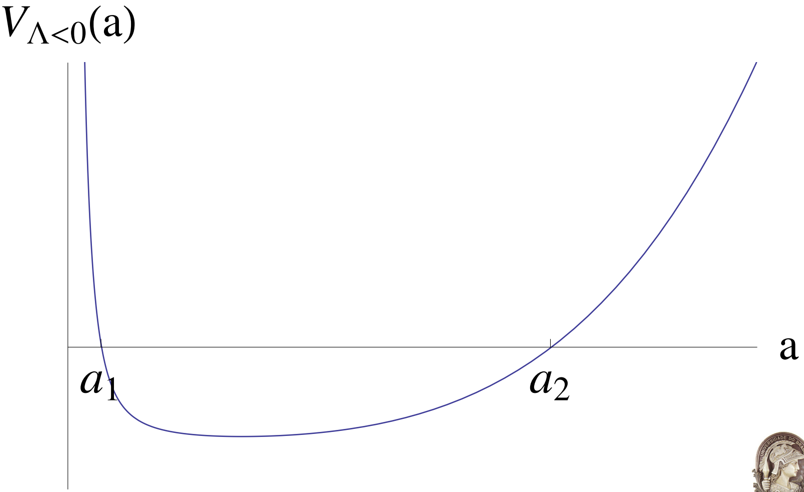


Figure: Potential for $\Lambda < 0$

Hořava-Lifshitz minisuperspace potentials: $\Lambda > 0$

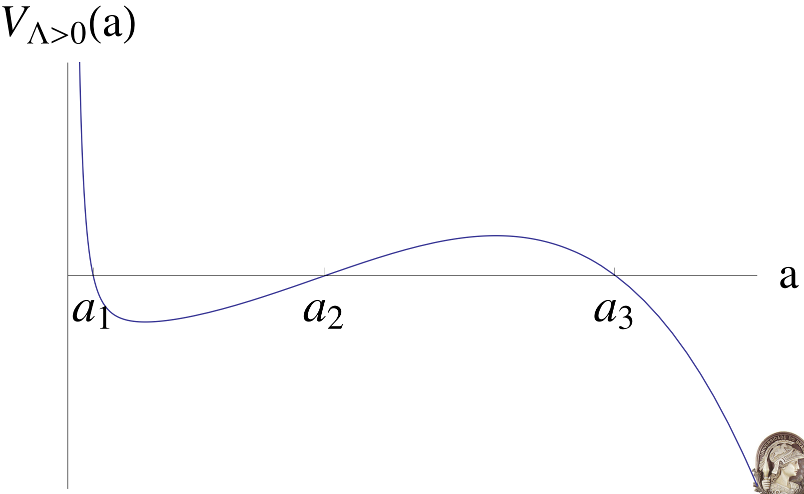


Figure: Potential for $\Lambda > 0$

3 -Solutions of the Wheeler-deWitt Equation

Boundary Conditions

- One chooses the deWitt boundary condition, which is expressed, for FLRW models as

$$\psi_{dW}(a=0)=0.$$

- Notice that this boundary condition does not mean that there is a quantum avoidance of the classical singularity.
- Indeed some examples were given
  - where $\psi \to 0$ but $\int da|\psi(a)|^2$ diverges
  - and conversely cases where $\int da|\psi(a)|^2 \to 0$ but ψ diverges.

WDW equation solution for $a \ll 1$

- This region corresponds to the very early Universe, where HL terms dominate. The WdW equation reads and the solution is:

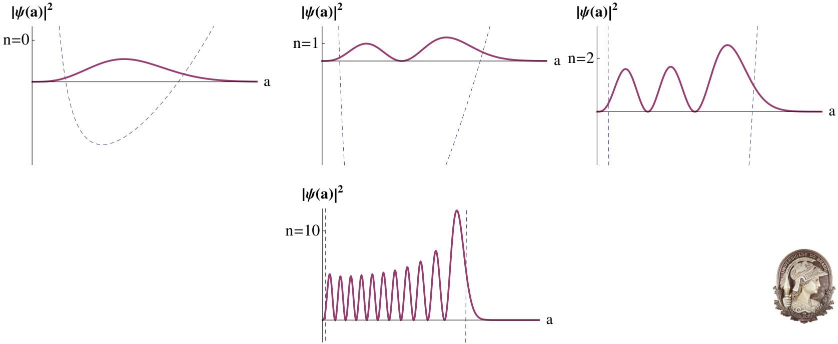
$$\left\{ \frac{d^2}{da^2} + \frac{g_s}{a^2} \right\} \psi(a) = 0 \quad \Rightarrow \quad \psi(a) \sim a^{1/2 + 1/2\sqrt{1-4g_s}}.$$

- A “quantum” bound for g_s is found, $i.e.$ $g_s \leq 1/4$.

WDW equation solution for $\Lambda = 0$

The normalised wave function is:

$$\psi(a) = \sqrt{\frac{2n! g_C^{1/4}}{\Gamma(n+\alpha+1)}} e^{-\frac{\sqrt{g_C} a^2}{2}} \left(g_C^{1/4} a\right)^{\alpha+\frac{1}{2}} L_n^{(\alpha)}(\sqrt{g_C} a^2).$$



WDW equation solution for $\Lambda \neq 0$

- If $g_\Lambda \neq 0$, WdW equation cannot be analitically solved.
- The behaviour for large a and nearby the singularity $a = 0$ were already discussed.
- For the intermediate regions where the curvature term starts to become relevant, one has to rely on the WKB approximation, which for the the classical allowed region is given by

$$\psi(a) \approx \frac{1}{|V(a)|^{1/4}} \exp\left[\pm i \int_{a_1}^{a} \sqrt{|V(a)|} da\right],$$

where the following integral must be solved:

$$\int_{a_1}^{a} \sqrt{|V(a)|} da = \sqrt{g_\Lambda} \int_{a_1}^{a} \frac{\sqrt{(a^2 - a_1^2)(a_2^2 - a^2)(a_3^2 - a^2)}}{a} da$$

- This integral can be written as a sum of elliptic integrals.

Conclusions

- Quantum cosmology applied to HL gravity suggests that this proposal matches the expectation of a quantum gravity model for the very early universe, as it provides, for instance, a hint for the singularity problem for the $\Lambda = 0$ case.
- In what concerns specific solutions, the model suggests that the GR behaviour is recovered at the semiclassical limit.