Gravitational Waves and New Perspectives for Quantum Gravity

Ilya L. Shapiro

Universidade Federal de Juiz de Fora, Minas Gerais, Brazil

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Contents

- **Why should we quantize gravity.**
Semiclassical approach and higher derivatives.

- **Power counting. Quantum GR or Higher derivative QG?**
Ghosts in renormalizable and superrenormalizable QG.

- **Why people do not like ghosts? Do they pose a danger?**

- **Gravitational waves and stability of classical solutions.**

General Relativity

(GR) is a complete theory of classical gravitational phenomena. It proved valid in the wide range of energies and distances.

There are covariant equations for the matter (fields and particles, fluids etc) and Einstein equations for the gravitational field $g_{\mu\nu}$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G\, T_{\mu\nu} - \Lambda\, g_{\mu\nu}\,.$$

The most important solutions of GR have specific symmetries.

1) Spherically-symmetric solution. Stars ... Black holes.
2) Isotropic and homogeneous metric. Universe.

Both cases are characterized by singularities, when curvature and density of matter become infinite.

GR is not valid at all scales.

Dimensional arguments.

The expected scale of the quantum gravity effects is associated to the Planck units of length, time and mass. The idea of Planck units is based on the existence of the 3 fundamental constants:

$$c = 3 \cdot 10^{10}\, cm/s\,,$$

$$\hbar = 1.054 \cdot 10^{-27}\, erg \cdot sec\,;$$

$$G = 6.67 \cdot 10^{-8}\, cm^3/sec^2\, g\,.$$

One can use them uniquely to construct the dimensions of

length $l_P = G^{1/2}\,\hbar^{1/2}\,c^{-3/2} \approx 1.4 \cdot 10^{-33}\, cm;$

time $t_P = G^{1/2}\,\hbar^{1/2}\,c^{-5/2} \approx 0.7 \cdot 10^{-43}\, sec;$

mass $M_P = G^{-1/2}\,\hbar^{1/2}\,c^{1/2} \approx 0.2 \cdot 10^{-5} g \approx 10^{19}\, GeV.$

Three choice for Quantum Gravity (QG)

One may suppose that the fundamental units indicate to the presence of fundamental physics at the Planck scale.

We can classify the possible approaches into three distinct groups. Namely, we can

● Quantize both gravity and matter fields. This is the most fundamental approach and the main subject of this talk.

●● Quantize only matter fields on classical curved background (semiclassical approach).
QFT and Curved space-time are well-established notions, which passed many experimental/observational tests.

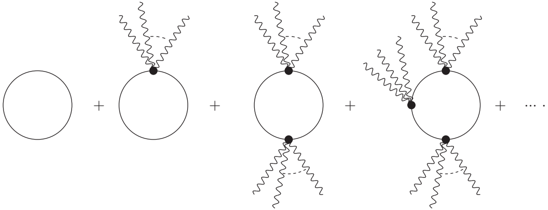
●●● Quantize something else. E.g., in case of (super)string theory both matter and gravity are induced.

●● **The renormalizable QFT in curved space requires introducing a generalized action of gravity (external field).**

It is easy to see that the theory can be renormalizable only if we include higher derivative terms into vacuum action.

Introduction: *Buchbinder, Odintsov & I.Sh. (1992).*

Relevant diagrams for the vacuum sector



All possible covariant counterterms have the structure of

$$S_{vac} = S_{EH} + S_{HD}$$

The necessary form of the “vacuum action” is as follows:

$$S_{vac} = S_{EH} + S_{HD} \,,$$

where $$S_{EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,\{R + 2\Lambda\}\,.$$

is the Einstein-Hilbert action with the cosmological constant.

S_{HD} **includes higher derivative terms. The most useful form is**

$$S_{HD} = \int d^4x\sqrt{-g}\,\{a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2\}\,,$$

where $$C^2(4) = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + 1/3\,R^2$$

is the square of the Weyl tensor in $d = 4$ **and**

$$E = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4\,R_{\alpha\beta}R^{\alpha\beta} + R^2$$

is integrand of the Gauss-Bonnet term (topological term in d=4**).**

Observation about higher derivatives.

A consistent theory of quantum matter fields on classical curved background can be achieved only if we include

$$S_{HD} = \int d^4x \sqrt{-g}\, \left\{ a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2 \right\},$$

along with Einstein-Hilbert and cosmological constant terms.

In quantum gravity such a higher derivative (HD) term means massive ghost, a spin-two particle with negative kinetic energy. This leads to the problem with unitarity, at least at the tree level.

But, in the semiclassical theory gravity is external and unitarity of the gravitational S-matrix is not really important.

The consistency conditions in this case include:

(i) **existence of physically reasonable solutions and**
(ii) **their stability under small metric perturbations.**

Further arguments:

- **One should definitely quantize both matter and gravity, for otherwise the QG theory would not be complete.**

- **The diagrams with matter internal lines in a complete QG are be exactly the same as in a semiclassical theory.**

- **This means one can not quantize metric without higher derivative terms in a consistent manner, since these terms are produced already in the semiclassical theory.**

- **Indeed, many important results in curved-space QFT are related to the renormalization of higher derivative vacuum terms, including Hawking radiation, Starobinsky inflation and others.**

How we quantize gravity?

The standard "old" QG starts from the following bootstrap:

- **Quantization can be performed using the variables**

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,. \tag{1}$$

It is assumed that this choice of the quantum metric is the "right" one. This choice enables one to deal with the well-defined object such as S **- matrix.**

- **In case of quantum GR** (1) **corresponds to so-called Gaussian expansion of the action. We can take**

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}\,. \tag{2}$$

and then, for

$$S_{EH} = -\frac{1}{\kappa^2}\int d^4x\sqrt{-g}\,R\,, \qquad \kappa^2 = 16\pi G \tag{3}$$

this means we have the second order (Gaussian approximation) of the action to be κ**- independent.**

- **Finally, the expansion around the flat background means we have Newton constant as a coupling.**

- **In the recent literature one can find a lot of work about possible “non-Gaussian fixed point” in QG. Such a fixed point was originally conjectured by S. Weinberg as a hypothesis of “asymptotic safety” – a weaker alternative to renormalizability.**

- **If being confirmed, then we do not need to construct the quantum theory around flat space. Instead, the theory should be constructed with some other background in mind.**

- **But then, what is the level of importance of the** S **- matrix approach, becomes very unclear.**

- **This is not clear, in general, for QG.**

Quantum Gravity (QG)

starts from some covariant action of gravity,

$$S = \int d^4x \sqrt{-g}\, \mathcal{L}(g_{\mu\nu})\,.$$

$\mathcal{L}(g_{\mu\nu})$ **can be of GR,** $\mathcal{L}(g_{\mu\nu}) = -\kappa^{-2}(R + 2\Lambda)$ **or some other.**

Gauge transformation $x'^{\mu} = x^{\mu} + \xi^{\mu}$. **The metric transforms as**

$$\delta g_{\mu\nu} = g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -\nabla_{\mu}\xi_{\nu} - \nabla_{\nu}\xi_{\mu}\,.$$

In the case of $\quad g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x)\,,$

$$\delta h_{\mu\nu} = -\frac{1}{\kappa}\,(\partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu}) - h_{\mu\alpha}\partial_{\nu}\xi^{\alpha} - h_{\nu\alpha}\partial_{\mu}\xi^{\alpha} - \xi^{\alpha}\partial_{\alpha}h_{\mu\nu} = R_{\mu\nu\,,\,\alpha}\xi^{\alpha}\,.$$

The gauge invariance of the action means

$$\frac{\delta S}{\delta h_{\mu\nu}} \cdot R_{\mu\nu\,,\,\alpha}\,\xi^{\alpha} = 0\,.$$

Let us start from the Faddeev-Popov approach,

$$Z(J) = \int DhDCD\bar{C}\ \mathrm{Det}\,(Y_{\alpha\beta})$$

$$\times\ \exp\Big\{iS(h) + \frac{i}{2}\chi^\alpha\, Y_{\alpha\beta}\,\chi^\beta + \frac{i}{2}\bar{C}^\alpha\, M^\beta_\alpha\, C^\beta + iJ^{\mu\nu}h_{\mu\nu}\Big\}\,.$$

where the ghost part is $\quad M^\beta_\alpha \,=\, \dfrac{\delta\chi^\alpha}{\delta h_{\mu\nu}}\, R_{\mu\nu,\alpha}\,.$

The useful choice of the gauge fixing condition and the weight function depends on the theory.

The most popular gauges are the Fock-deDonder one

$$\chi^\mu = \partial_\nu \Phi^{\mu\nu}\,, \qquad \kappa\Phi^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}\,,$$

harmonic gauge $\quad \chi_\mu = \partial^\nu h_{\mu\nu} - \beta\partial_\mu h\,, \qquad \beta = \dfrac{1}{2}\,,$

and background gauges

$\chi_\mu = \nabla^\nu h_{\mu\nu} - \beta\nabla_\mu h\,,$ **where** $g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}\,.$

Consider the total action with $g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x)$ **,**

$$S_t \;=\; S(h) + \frac{1}{2}\chi^\alpha\, Y_{\alpha\beta}\, \chi^\beta + \frac{1}{2}\bar{C}^\alpha\, M^\beta_\alpha\, C^\beta\,.$$

It is not gauge invariant, instead there is BRST invariance,

$$\delta_{BRST} h_{\mu\nu} = \kappa R_{\mu\nu\,,\,\alpha}\, C^\alpha\, \delta\mu\,,$$

$$\delta_{BRST} C^\alpha = \frac{1}{2} f^\alpha{}_{\beta\gamma}\, C^\beta\, C^\gamma\, \delta\mu \;=\; C^\beta\, \partial_\alpha C^\alpha \cdot \delta\mu\,.$$

$$\delta_{BRST}\bar{C}_\alpha = Y_{\alpha\beta}\,\chi^\beta(h)\,\delta\mu\,,$$

It proves useful introducing two extra sources. The total action

$$\bar{S} \;=\; S_t + K^{\mu\nu} R_{\mu\nu\,,\,\alpha}\bar{C}^\alpha + L_\sigma \partial_\beta\, C^\sigma\, C^\beta\,.$$

One can easily prove that it satisfies the Noether identity

$$\frac{\delta\bar{S}}{\delta K^{\mu\nu}}\,\frac{\delta\bar{S}}{\delta h_{\mu\nu}} \;+\; \frac{\delta\bar{S}}{\delta L_\sigma}\,\frac{\delta\bar{S}}{\delta\bar{C}^\sigma} \;+\; Y_{\alpha\beta}\,\chi^\beta\,\frac{\delta\bar{S}}{\delta\bar{C}_\alpha} \;=\; 0\,.$$

One performs the BRST transformation in Z and since it does not change, arrive at the Slavnov-Taylor identities for Z, where

$$Z(J,\bar{\beta},\beta,K,L) \,=\, \int DhDCD\bar{C}\;\mathrm{Det}\,(Y_{\alpha\beta})$$

$$\times\;\exp\Big\{i\bar{S} + iJ^{\mu\nu}h_{\mu\nu} + i\bar{C}_\mu\beta^\mu + i\bar{\beta}_\mu C\mu\Big\}\,.$$

As a next step we define $W = -i\ln Z$ and the mean fields

$$\langle h_{\mu\nu}\rangle = \frac{\delta W}{\delta J^{\mu\nu}}\,, \qquad \langle C^\sigma\rangle = \frac{\delta W}{\delta\bar{\beta}_\sigma}\,, \qquad \langle\bar{C}_\rho\rangle = \frac{\delta W}{\delta\beta^\rho}\,.$$

and the corresponding (usefully modified) effective action

$$\tilde{\Gamma}(h_{\mu\nu},\,C^\sigma,\,\bar{C}_\rho,\,K^{\mu\nu},\,L_\sigma) \,=\, W(J^{\mu\nu},\,\bar{\beta}_\sigma,\,\beta^\rho,\,K^{\mu\nu},\,L_\sigma)$$

$$-h_{\mu\nu}J^{\mu\nu} - \bar{C}_\rho\beta^\rho - C^\sigma\,\bar{\beta}^\sigma - \frac{1}{2}\,\chi^\alpha\,Y_{\alpha\beta}\,\chi^\beta\,.$$

One can prove that it satisfies the identities

$$\frac{\delta\tilde{\Gamma}}{\delta K^{\mu\nu}}\,\frac{\delta\tilde{\Gamma}}{\delta h_{\mu\nu}} + \frac{\delta\tilde{\Gamma}}{\delta L_\sigma}\,\frac{\delta\tilde{\Gamma}}{\delta\bar{C}^\sigma} \,=\, 0\,; \qquad \frac{\delta\chi^\alpha}{\delta h_{\rho\sigma}}\,\frac{\delta\tilde{\Gamma}}{\delta K^{\rho\sigma}} - \frac{\delta\tilde{\Gamma}}{\delta\bar{C}_\alpha} \,=\, 0\,.$$

Consider the loop expansion

$$\tilde{\Gamma} = \tilde{S} + \sum_{k=1}^{\infty} \hbar^k \, \tilde{\Gamma}^{(k)}\,, \quad \text{where} \quad \tilde{\Gamma}^{(k)} = \tilde{\Gamma}^{(k)}_{div} + \tilde{\Gamma}^{(k)}_{fin}\,.$$

One can prove that general local solution for divergent part is

$$\tilde{\Gamma}^{(k)}_{div} = A(h_{\mu\nu}) + \mathcal{E}X(h_{\mu\nu}, C^{\sigma}, \bar{C}_{\rho}, K^{\mu\nu}, L_{\sigma})\,,$$

where A **is some covariant functional and**
X **is an arbitrary local functional of its variables.**

The $\mathcal{E}X$ **- terms can be removed by**

- **Renormalization of the field** $h_{\mu\nu}$ **together with some gauge transformation.**
- **Renormalization of the FP ghosts** C_{μ} **and** $\bar{C}^{\nu}$ **together with some BRST transformation.**

Finally, the problem is reduced to the possible form of the covariant local functional $A(h_{\mu\nu})$.

Let us use the notion of power counting to explore $A(h_{\mu\nu})$. The universal formula for the superficial degree of divergence is

$$D + d \, = \, \sum_{l_{int}} (4 - r_l) \, - \, 4n \, + \, 4 \, + \, \sum_{\nu} K_\nu \, .$$

Here

D is the superficial degree of divergence for a given diagram,
d is the number of derivatives on external lines of the diagram,
r_l is the power of momenta in the propagator of internal line,
n is the number of vertices and
K_ν is the power of momenta in a given vertex.

On the top of that one can use topological relation between number of loops p, vertices n, and internal lines

$$l_{int} \, = \, p + n - 1 \, .$$

As the first example consider quantum GR.

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R+2\Lambda)\,.$$

For the sake of simplicity we consider only vertices with maximal K_ν. **Then we have** $r_l = K_\nu = 2$ **and, combining**

$$D+d = \sum_{l_{int}}(4-r_l) - 4n + 4 + \sum_\nu K_\nu$$

with

$$l_{int} = p+n-1$$

we arrive at the estimate ($D=0$ **means log. divergences)**

$$D+d = 2+2p\,.$$

This output means that quantum GR is not renormalizable.

More details: What means the relation

$$D + d \;=\; 2 + 2p \quad ?$$

Remember that $D = 0$ means logarithmic divergences.

At the 1-loop level we can expect the divergences like

$$\mathcal{O}(R^2_{...}) = R^2_{\mu\nu\alpha\beta}\,,\;\; R^2_{\mu\nu}\,,\;\; R^2\,.$$

t'Hooft and Veltman; Deser and van Nieuwenhuisen, (1974); ...

At the 2-loop level we have

$$\mathcal{O}(R^3_{...}) = R_{\mu\nu}\Box R^{\mu\nu}\,,\; ...R^3\,,\;\; R_{\mu\nu}R^{\mu}_{\alpha}R^{\alpha\nu}\,,\;\; R_{\mu\nu\alpha\beta}R^{\mu\nu}{}_{\rho\sigma}R^{\mu\nu\rho\sigma}\,.$$

Goroff and Sagnotti, (1986).

The last structure does not vanish on-shell and this proves that the theory is not renormalizable, at least within the standard perturbative approach.

Within the standard perturbative approach non-renormalizability means the theory has no predictive power.

Every time we introduce a new type of a counterterm, it is necessary to fix renormalization condition and this means a measurement. So, before making a single predictions, it is necessary to have an infinite amount of experimental data.

What are the possible solutions?

- **Change standard perturbative approach to something else.**

There are many options, but their consistency or their relation to the QG program are not clear, in all cases.

- **Change the theory, i.e., take another theory to construct QG.**

The first option is widely explores in the asymptotic safety scenarios, in the effective approaches to QG, induced gravity approach (including string theory) and so on.

Let us concentrate on the second idea.

Perhaps, the most natural is HDQG.

Reason: we need HD's anyway for quantum matter field.

Already known action: $S_{gravity} = S_{EH} + S_{HD}$

where

$$S_{EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,\{R + 2\Lambda\}$$

and S_{HD} **include higher derivative terms**

$$S_{HD} = -\int d^4x\sqrt{-g}\left\{\frac{1}{2\lambda}C^2 + \frac{1}{\rho}E + \tau\Box R + \frac{\omega}{3\lambda}R^2\right\},$$

$$C^2(4) = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + 1/3\,R^2,$$

$$E = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4\,R_{\alpha\beta}R^{\alpha\beta} + R^2.$$

K. Stelle, Phys. Rev. D (1977).

Buchbinder, Odintsov, I.Sh., Effective Action in Quantum Gravity. (1992 - IOPP).

Propagators and vertices in HDQG are not like in quantum GR.
Propagators of metric and ghosts behave like $\mathcal{O}(k^{-4})$ and we have K_4, K_2, K_0 vertices. The superficial degree of divergence

$$D + d = 4 - 2K_2 - 2K_0.$$

This theory is definitely renormalizable. Dimensions of counterterms are 4, 2, 0.

Well, there is a price to pay: Massive ghosts

$$G_{\rm spin-2}(k) \sim \frac{1}{m^2}\left(\frac{1}{k^2} - \frac{1}{m^2+k^2}\right), \quad m \propto M_P\,.$$

The tree-level spectrum includes massless graviton and massive spin-2 "ghost" with negative kinetic energy and huge mass.

The main point of this talk is a new proposal concerning ghosts and related difficulty of QG.

Including even more derivatives was initially thought to move massive pole to even higher mass scale,

$$S = S_{EH} + \int d^4x\sqrt{-g}\,\Big\{a_1 R^2_{\mu\nu\alpha\beta} + a_2 R^2_{\mu\nu} + a_3 R^2 + \,...$$

$$+\, c_1 R_{\mu\nu\alpha\beta}\Box^k R^{\mu\nu\alpha\beta} + c_2 R_{\mu\nu}\Box^k R^{\mu\nu} + c_3 R\Box^k R + \,b_{1,2,..} R^{k+1}_{...}\Big\}\,.$$

Simple analysis shows this theory is superrenormalizable, but the massive ghosts are still here. For the case of real poles:

$$G_2(k) = \frac{A_0}{k^2} + \frac{A_1}{k^2+m_1^2} + \frac{A_2}{k^2+m_2^2} + \cdots + \frac{A_{N+1}}{k^2+m_{N+1}^2}$$

For any sequence $0 < m_1^2 < m_2^2 < m_3^2 < \cdots < m_{N+1}^2$, **the signs of the corresponding terms alternate,** $A_j \cdot A_{j+1} < 0$.

Asorey, Lopez & I. Sh., hep-th/9610006; IJMPhA (1997).

$$S = \int d^4x\sqrt{-g}\,\Big\{c_1 R_{\mu\nu\alpha\beta}\Box^k R^{\mu\nu\alpha\beta} + c_2 R_{\mu\nu}\Box^k R^{\mu\nu} + c_3 R\Box^k R + \,...\Big\}\,.$$

Again, let us consider only vertices with a maximal $K_\nu = 2k+4$.

Then we have $r_l = K_\nu = 2k+4$ **and, combining**

$$D + d \;=\; \sum_{l_{int}} (4 - r_l) \,-\, 4n \,+\, 4 \,+\, \sum_\nu K_\nu$$

with $\qquad l_{int} = p + n - 1\,,$

we can easily arrive at the estimate of d for $D = 0$

$$d \;=\; 4 \,+\, k(1-p)\,.$$

For $k=0$ **we meet the standard HDQG result,** $d \equiv 4$. **Starting from** $k=1$ **we have superrenormalizable theory, where the divergences exist only for** $p = 1, 2, 3$.

For $k \geq 3$ **we have superrenormalizable theory, where the divergences exist only for** $p = 1$, **that is at the one-loop level.**

Interesting features of HDQG

- **If we solve the ghost problem someday, there will be many versions of renormalizable and superrenormalizable HDQG's.**

- **In the basic fourth-derivative version the quantum corrections have an ambiguity related to the choice of gauge-fixing condition. E.g., there is no well-defined β- function for G and cosmological constant Λ .**

- **In the superrenormalizable versions there is no such problem, in fact all β- functions are well-defined.**
- **For $k \geq 3$ it is not impossible to derive exact β-functions.**

- **However, we will meet quite a lot of theories with many free parameters, and still not a single experiment to fix them. Obviously, the main problem of QG is not a theory!**

Once again: what is bad in the higher-derivative gravity?

For the linearized gravity

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{1}$$

we meet

$$G_{\text{spin}-2}(k) \sim \frac{1}{m^2}\left(\frac{1}{k^2} - \frac{1}{m^2+k^2}\right), \quad m \propto M_P\,.$$

This means that the tree-level spectrum includes massless graviton and massive spin-2 "ghost" with negative kinetic energy and huge mass.

- **Interaction between ghost and gravitons may violate energy conservation in the massless sector** *(M.J.G. Veltman, 1963).*

- **In classical systems higher derivatives generate exploding instabilities at the non-linear level** *(M.V. Ostrogradsky, 1850).*

- **Without ghost one violates unitarity of the S -matrix.**

There were several attempts to solve the HD ghost problem.

The mainstream approach:

Stelle, Salam & Strathdee, Tomboulis,
Antonidis & Tomboulis, Johnston, Hawking, ...

Resummation of the perturbative series & expectation that the ghost becomes unstable and disappears in the *out* state.

The realization of this idea requires that
(i) Loop corrections shift ghost pole to the complex plane;
(ii) The position of the ghost pole gets gauge-fixing dependent.

With these assumptions one can prove the unitarity of the theory.

Unfortunately, the existing perturbative and non-perturbative (e.g., $1/N$ expansion, lattice formulations etc) methods are not sufficient to claim whether these two conditions or at least part of them are satisfied or not.

An alternative idea:

S.W. Hawking, Who's Afraid Of (higher Derivative) Ghosts? (1985);
S.W. Hawking, Th. Hertog, Living with ghosts, Phys.Rev. D65 (2002); hep-th/0107088.

An attempt to consider ghost and graviton together at the non-linear level in quantum theory.

The realization of this interesting idea requires qualitatively new formulation of QFT.

Does at least one of these approaches really work?

For a while, there is no definite answer.

In what follows we suggest a new approach which is much simpler and is probably working.

Assumptions we made to condemn higher derivative theory:

● One can draw conclusions using linearized gravity approximation. S-matrix of gravitons is the main object.

●● Ostrogradsky instabilities or Veltman scattering are relevant independent on the energy scale, in all cases they produce run-away solutions and "Universe explodes".

There is a simple way to check all these assumptions at once.

Let us take a higher derivative theory of gravity and verify the stability with respect to the linear perturbations on some, physically interesting, dynamical background.

If assumptions are correct, we observe rapidly growing modes even for the low-energy (e.g., low-curvature) background.

However, if there are no growing modes at the linear level, there will not be such modes even at higher orders!

Remember that the ghost problem is a tree-level one!

For the sake of generality, consider not only classical HD terms, but also take into account the semiclassical corrections, derived by integrating conformal anomaly.

● **Consider the theory with higher derivative terms plus anomaly-induced effective action of gravity.**

● ● **Find an approximate solution without higher derivative terms and quantum corrections.**

● ● ● **Derive gravitational waves in the (quantum-corrected or not) theory and explore the stability of classical solutions.**

Program realized for the cosmological background in
Fabris, Pelinson, Salles & I.Sh., JCAP 02 (2012); 1112.5202 [gr-qc].
F. Salles & I.Sh., PRD (2014); 1401.4583 [hep-th].

Semiclassical theory of classically conformal free fields.

The vacuum action in the general case has the form

$$S_{vac} \,=\, S_{EH} \,+\, S_{HD}\,,$$

$$S_{HD} = \int d^4x\sqrt{-g}\Big\{\alpha C^2 + \beta E + \gamma\Box R + \delta R^2\Big\}\,,$$

where

$$C^2 = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + \frac{1}{3}\,R^2\,, \qquad E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\alpha\beta} + R^2\,.$$

At quantum level $S_{vac}(g_{\mu\nu})$ **is replaced by effective action (EA)**

$$\Gamma_{vac}(g_{\mu\nu}) \,=\, S_{vac}(g_{\mu\nu}) \,+\, \bar{\Gamma}_{vac}(g_{\mu\nu})\,.$$

The anomalous trace is

$$\langle T^\mu_\mu \rangle \,=\, \left\{ \beta_1 C^2 + \beta_2 E + a'\Box R \right\}\,, \quad \textbf{where}$$

$$\left(\begin{array}{c} \omega \\ -b \\ c \end{array} \right) = \left(\begin{array}{c} \beta_1 \\ -\beta_2 \\ \beta_3 \end{array} \right) = \frac{1}{360(4\pi)^2} \left(\begin{array}{c} 3N_0 + 18N_{1/2} + 36N_1 \\ N_0 + 11N_{1/2} + 62N_1 \\ 2N_0 + 12N_{1/2} - 36N_1 \end{array} \right)$$

N_0 **conformal real scalars**, $N_{1/2}$ **Dirac spinors**, N_1 **vectors**.

One can use $\langle T^\mu_\mu \rangle$ to find the finite part of 1-loop EA

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu} \frac{\delta\, \Gamma_{ind}}{\delta g_{\mu\nu}} \,=\, \langle T^\mu_\mu \rangle \,=\, \frac{1}{(4\pi)^2}\, \left(\omega C^2 + bE + c\Box R \right)\,.$$

Solution *(Riegert, Fradkin & Tseytlin, PLB-1984).*

Useful notation:

$$\Delta = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}\,R\Box + \frac{1}{3}\,(\nabla^\mu R)\nabla_\mu\,.$$

Anomaly-induced effective action of vacuum

$$\Gamma_{ind} = S_c[g_{\mu\nu}] - \frac{3c+2b}{36(4\pi)^2} \int d^4x\sqrt{-g(x)}\,R^2(x)$$

$$+ \frac{\omega}{4} \int_x \int_y C^2(x)\,G(x,y)\,\big(E - \frac{2}{3}\Box R\big)_y$$

$$+ \frac{b}{8} \int_x \int_y \big(E - \frac{2}{3}\Box R\big)_x\,G(x,y)\,\big(E - \frac{2}{3}\Box R\big)_y\,,$$

where $\int_x = \int d^4x\sqrt{-g}\,, \qquad \Delta_4\,G(x,y) = \delta(x,y)$.

One can rewrite this expression using auxiliary scalars,

$$\Gamma_{ind} \;=\; S_c[g_{\mu\nu}] - \frac{3c+2b}{36(4\pi)^2} \int_x R^2(x) + \int_x \Big\{ \frac{1}{2}\,\varphi\Delta_4\varphi - \frac{1}{2}\,\psi\Delta_4\psi$$

$$+ \frac{a}{8\pi\sqrt{-b}}\,\psi\,C^2 + \varphi\,\Big[\frac{\sqrt{-b}}{8\pi}\,\big(E - \frac{2}{3}\,\Box R\big) - \frac{a}{8\pi\sqrt{-b}}\,C^2 \Big] \Big\}\,.$$

where $S_c[\bar{g}_{\mu\nu}] = S_c[g_{\mu\nu}]$ **is an integration constant.**

An important application is the Modified Starobinsky Model

Fabris, Pelinson, Solà, I.Sh. et al. (1998-2003, 2011-2012).

Consider a Cosmological Model based on the action

$$S_{total} \,=\, -\,\frac{M_P^2}{16\pi}\,\int d^4x\sqrt{-g}\,(R + 2\Lambda) + S_{matter} + S_{vac} + \bar{\Gamma}_{ind}\,.$$

Equation of motion in physical time $dt = a(\eta)d\eta$

$$\frac{\ddot{\ddot{a}}}{a} + \frac{3\dot{a}\dot{\ddot{a}}}{a^2} + \frac{\ddot{a}^2}{a^2} - \left(5 + \frac{4b}{c}\right)\frac{\ddot{a}\dot{a}^2}{a^3} - 2k\left(1 + \frac{2b}{c}\right)\frac{\ddot{a}}{a^3}$$

$$-\frac{M_P^2}{8\pi c}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{2\Lambda}{3}\right) \,=\, 0\,,$$

where $k = 0, \pm 1,\ \Lambda$ **is the cosmological constant.**

Particular solutions *(Starobinsky, PLB-1980)*

$$a(t) = \begin{cases} a_o \exp(Ht)\,, & k=0 \\ a_o \cosh(Ht)\,, & k=1 \\ a_o \sinh(Ht)\,, & k=-1 \end{cases} ,$$

Hubble parameter $$H = \frac{M_P}{\sqrt{-32\pi b}} \left(1 \pm \sqrt{1 + \frac{64\pi b}{3} \frac{\Lambda}{M_P^2}} \right)^{1/2} .$$

Perturbations of the conformal factor:

$$\sigma(t) \to \sigma(t) + y(t).$$

The criterion for a stable inflation

$$c > 0 \quad \Longleftrightarrow \quad N_1 < \frac{1}{3} N_{1/2} + \frac{1}{18} N_0 \,,$$

in agreement to Starobinsky (1980).

Remark. **The value of Λ and the choice of $k = 0, \pm 1$ do not influence the stability condition.**

Unstable case:

$$N_1 > \frac{1}{3} N_{1/2} + \frac{1}{18} N_0 \,.$$

One can fine-tune initial conditions such that the Universe has sufficient inflation and graceful exit to the FRW behaviour.

The Starobinsky model is based on the unstable case

M.V. Fischetti, J.B. Hartle & B.L. Hu, PRD20 (1979);

- *A.A. Starobinski, Ph.L.B 91 (1980) 99;*

V.F. Mukhanov & G.V. Chibisov, JETP Lett. 33 (1981); JETP (1982);
A.A.Starobinski, Let.Astr.Journ. 9 (1983);
A. Vilenkin, PRD 32 (1985);
P. Anderson, PRD (1983-85).

Stable case: The Universe starts inflation from an arbitrary position at the phase plane, anomaly-induced inflation "kills" any matter content in a few dozens of Planck times.

Pelinson, I.Sh., Takakura, NPB; NPB (PS) - 2003.

Simple test of the Modified Starobinsky Model.

Pelinson, I.Sh. & Takakura IRGA-NPB(PS)- 2003.,

Consider late Universe, $k = 0, \; H_0 = \sqrt{\Lambda/3}$.

Only photon is active, $N_0 = 0\,, \quad N_{1/2} = 0\,, \quad N_1 = 1\,.$

Graviton typical energy is $H_0 \approx 10^{-42}\,GeV$, $\Longrightarrow$ **all massive particles (even neutrino)** $m_\nu \geq 10^{-12}\,GeV$ **decouple from gravity.** $c < 0 \quad \Longrightarrow$ **today inflation is unstable.**

Stability for the small $H = H_0$ **case:** $H \to H_0 + const \cdot e^{\lambda t}$

$$\lambda^3 + 7H_0\lambda^2 + \left[\frac{(3c-b)4{H_0}^2}{c} - \frac{M_P^2}{8\pi c}\right]\lambda - \frac{32\pi b{H_0}^3 + M_P^2 H_0}{2\pi c} = 0\,.$$

The solutions are $\lambda_1 = -4H_0\,, \quad \lambda_{2/3} = -\frac{3}{2}H_0 \pm \frac{M_P}{\sqrt{8\pi|c|}}\,i\,.$

$\Lambda > 0$ **efficiently protects low-energy solution from higher-derivative instabilities in this case.**

Consider unstable inflation, matter (or radiation) dominated Universe and assume that the Universe is close to the classical FRW solution. The equation is

$$\frac{\ddddot{a}}{a} + \frac{3\dot{a}\dddot{a}}{a^2} + \frac{\ddot{a}^2}{a^2} - \left(5 + \frac{4b}{c}\right)\frac{\ddot{a}\dot{a}^2}{a^3} - 2k\left(1 + \frac{2b}{c}\right)\frac{\ddot{a}}{a^3}$$

$$-\frac{M_P^2}{8\pi c}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{2\Lambda}{3}\right) = -\frac{1}{3c}\,\rho_{matter}\ ,$$

The terms of the first line, of quantum origin, behave like $1/t^4$.

The second line terms, of classical origin, behave like $1/t^2$.

After certain time the "quantum" terms become negligible.

Conclusion: For the dynamics of conformal factor, classical solutions are very good low-energy approximations in the theory with quantum corrections and/or higher derivatives.

Stability & Gravitational Waves

As far as classical action and quantum, anomaly-induced term, both have higher derivatives, an important question is whether the stability of classical solutions in cosmology holds or not.

Consider small perturbation

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}\,, \quad h_{\mu\nu} = \delta\, g_{\mu\nu}\,,$$

where $g^0_{\mu\nu} = \{1,\; -\delta_{ij}\, a^2(t)\}$, $\mu = 0,1,2,3$ **and** $i = 1,2,3$.

$$h_{\mu\nu}(t,\vec{r}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{r}\cdot\vec{k}}\, h_{\mu\nu}(t,\vec{k})\,.$$

Using the conditions $\partial_i\, h^{ij} = 0$ **and** $h_{ii} = 0$, **together with the synchronous coordinate condition** $h_{\mu 0} = 0$, **we arrive at the equation for the tensor mode**

F. Salles and I.Sh., PRD (2014).

$$
\Big(2f_1+\frac{f_2}{2}\Big)\,\overset{....}{h}+\big[3H\,(4f_1+f_2)\big]\,\dddot{h}+\Big[3H^2\Big(6f_1+\frac{f_2}{2}-4f_3\Big)
$$

$$
+6\dot{H}\,(f_1-f_3)+\frac{1}{2}f_0\Big]\,\ddot{h}+\,(4f_1+f_2)\,\Big[\frac{\nabla^4 h}{4\,a^4}-\frac{\nabla^2\ddot{h}}{a^2}-H\,\frac{\nabla^2\dot{h}}{a^2}\Big]
$$

$$
+\Big[\frac{3}{2}Hf_0-21H\dot{H}\Big(\frac{1}{2}f_2+2f_3\Big)-\ddot{H}\Big(\frac{3}{2}f_2+6f_3\Big)-9H^3(f_2+4f_3)\Big]\,\dot{h}
$$

$$
+\Big[f_0(2\dot{H}+3H^2)-(36\dot{H}H^2+18\dot{H}^2+24H\ddot{H}+4\,\dddot{H}\,)(f_1+f_2+3f_3)\Big]\,h
$$

$$
+\Big[H^2(4f_1+4f_2+12f_3)+2\dot{H}\,(f_1+f_2+3f_3)-\frac{1}{2}\,f_0\Big]\,\frac{\nabla^2 h}{a^2}\;=\;0\,.
$$

It looks much simpler than the previously considered eqs. with semiclassical corrections,

Fabris, Pelinson and I.Sh., (2001);
Fabris, Pelinson, Salles and I.Sh., (2011).

$$
\Big(2f_1 + \frac{f_2}{2}\Big)\ \ddddot{h} + \Big[3H\Big(4f_1 + f_2\Big) + 4\dot{f}_1 + \dot{f}_2\Big]\ \dddot{h} + \Big[3H^2\Big(6f_1 + \frac{f_2}{2} - 4f_3\Big)
$$

$$
+H\Big(16\dot{f}_1 + \frac{9}{2}\dot{f}_2\Big) + 6\dot{H}\Big(f_1 - f_3\Big) + 2\ddot{f}_1 + \frac{1}{2}\big(\ddot{f}_2 + f_0 + f_4\ddot{\varphi}\big) + \frac{3}{2}f_4 H\dot{\varphi} - \frac{2}{3}f_5\dot{\varphi}^2\Big]\ \ddot{h}
$$

$$
-\Big(4f_1 + f_2\Big)\ \frac{\nabla^2\ddot{h}}{a^2} + \Big[\dot{H}\Big(4\dot{f}_1 - 6\dot{f}_3\Big) - 21H\dot{H}\Big(\frac{1}{2}f_2 + 2f_3\Big) - \ddot{H}\Big(\frac{3}{2}f_2 + 6f_3\Big)
$$

$$
+3H^2\Big(4\dot{f}_1 + \frac{1}{2}\dot{f}_2 - 4\dot{f}_3\Big) - 9H^3\big(f_2 + 4f_3\big) + H\Big(4\ddot{f}_1 + \frac{3}{2}\ddot{f}_2\Big) + \frac{3}{2}f_4\dot{\varphi}\Big(3H^2 + \dot{H}\Big)
$$

$$
+H\big(3f_4\ddot{\varphi} + \frac{3}{2}f_0 - 2f_5\dot{\varphi}^2\big) + \frac{1}{2}f_4\ \dddot{\varphi}\ - \frac{4}{3}f_5\dot{\varphi}\ddot{\varphi}\Big]\ \dot{h} - \Big[H\Big(4f_1 + f_2\Big) + 4\dot{f}_1 + \dot{f}_2\Big]\ \frac{\nabla^2\dot{h}}{a^2}
$$

$$
+\Big[5f_4 H\ \dddot{\varphi}\ + f_4\ \ddddot{\varphi}\ - \big(36\dot{H}H^2 + 18\dot{H}^2 + 24H\ddot{H} + 4\ \dddot{H}\ \big)\Big(f_1 + f_2 + 3f_3\Big)
$$

$$
-H\dot{H}\Big(32\dot{f}_1 + 36\dot{f}_2 + 120\dot{f}_3\Big) - 8\ddot{H}\Big(\dot{f}_1 + \dot{f}_2 + 3\dot{f}_3\Big) - H^2\Big(4\ddot{f}_1 + 6\ddot{f}_2 + 24\ddot{f}_3\Big)
$$

$$
-4\dot{H}\Big(\ddot{f}_1 + \ddot{f}_2 + 3\ddot{f}_3\Big) - 9f_4\dot{\varphi}\Big(H^3 + H\dot{H}\Big) + f_4\ddot{\varphi}\Big(3H^2 + 5\dot{H}\Big) - H^3\Big(8\dot{f}_1 + 12\dot{f}_2 + 48\dot{f}_3\Big)
$$

$$+f_5\dot{\varphi}^2\Big(\frac{1}{2}H^2+\frac{1}{3}\dot{H}\Big)+\frac{2}{3}f_5H\dot{\varphi}\ddot{\varphi}-\frac{1}{6}f_5\ddot{\varphi}^2+\frac{1}{3}f_5\dot{\varphi}\,\dddot{\varphi}\,\Big]\,h+f_0\Big[2\dot{H}+3H^2\Big]\,h$$

$$+\Big[H\Big(2\dot{f}_1+\frac{1}{2}\dot{f}_2\Big)+2\dot{H}\Big(f_1+f_2+3f_3\Big)$$

$$-\frac{1}{2}\left(\ddot{f}_2+f_4\ddot{\varphi}+f_0+3f_4H\dot{\varphi}\right)-\frac{1}{3}f_5\dot{\varphi}^2\Big]\,\frac{\nabla^2h}{a^2}+\Big[2f_1+\frac{1}{2}f_2\Big]\,\frac{\nabla^4h}{a^4}\;=\;0\,,$$

where the f - terms are defined as

$$f_0\;=\;-\frac{{M_P}^2}{16\pi}\,;\qquad f_1\;=\;a_1+a_2-\frac{b+\omega}{2\sqrt{-b}}\,\varphi+\frac{\omega}{2\sqrt{-b}}\,\psi\,;$$

$$f_2\;=\;-2a_1-4a_2+\frac{\omega+2b}{\sqrt{-b}}\,\varphi-\frac{\omega}{\sqrt{-b}}\,\psi;$$

$$f_3\;=\;\frac{a_1}{3}+a_2-\frac{3c+2b}{36}-\frac{3b+\omega}{6\sqrt{-b}}\,\varphi+\frac{\omega}{6\sqrt{-b}}\,\psi;$$

$$f_4\;=\;-\frac{4\pi\sqrt{-b}}{3}\,;\qquad f_5\;=\;\frac{1}{2}\,.$$

Qualitative results were achieved by using

I. Analytical methods. We can approximately treat all coefficients as constants. There is a mathematically consistent way to check when (and whether) it works. With the Wolfram's Mathematica software, manipulating our equation is not so difficult, in fact.

II. Numerical methods. CMBEasy software or Wolfram's Mathematica 9 can be applied to gravitational waves and provide the results which are perfectly consistent with the output of method I.

Net Result: The stability does not actually depend on quantum corrections. It is completely defined by the sign of the classical coefficient a_1 of the Weyl-squared term. The sign of this term defines whether graviton or ghost has positive kinetic energy!

We can distinguish the following two cases:

- **The coefficient of the Weyl-squared term is** $a_1 < 0$ **Then**

$$G_{\rm spin-2}(k) \sim \frac{1}{m^2} \left(\frac{1}{k^2} - \frac{1}{m^2+k^2}\right), \quad m \propto M_P\,,$$

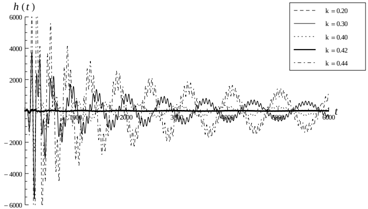
there are no growing modes up to the Planck scale, $\vec{k}^2 \approx M_P^2$**.**

For the dS background this is in a perfect agreement with
Starobinsky, Let.Astr.Journ. (in Russian) (1983);
Hawking, Hertog and Real, PRD (2001).

- **The classical coefficient of the Weyl-squared term** $a_1 > 0$**.**

$$G_{\rm spin-2}(k) \sim \frac{1}{m^2} \left(-\frac{1}{k^2} + \frac{1}{m^2+k^2}\right), \quad m \propto M_P\,.$$

and there are rapidly growing modes at any scale.



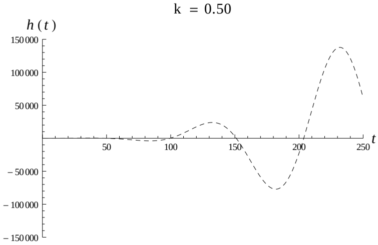


Illustration. Radiation-dominated Universe. There are no growing modes until the frequency k achieves the value ≈ 0.5 in Planck units. Starting from this value, we can observe the massive ghost making its destructive work.

So, where is the ghost??

In fact, the result is natural. The anomaly-induced quantum correction is $\mathcal{O}(R^3_{....})$ and $\mathcal{O}(R^4_{....})$, Until the energy is not of the Planck order of magnitude, these corrections can not compete with classical $\mathcal{O}(R^2_{....})$ - terms.

For $a_1 < 0$ there are no growing tensor modes in the higher derivative gravity on cosmological backgrounds.

Massive ghosts are present only in the vacuum state. We just do not observe them "alive" until the typical energy scale is below the Planck mass.

- All in all, massive ghosts do not pose real danger below the Planck scale. Above M_P we need new ideas to fight ghosts.

Other backgrounds?

- **Black hole solutions: conflicting results.**

B. Whitt, Phys. Rev. D32 (1985) 379.
Yu.S. Myung, Phys. Rev. D88 (2013) 024039, arXiv:1306.3725.

It is not clear yet to which extent the results depend on the choice of the boundary conditions, on the frequency of initial seeds of perturbations and on some technical assumptions done in these works.

The problem is rich in astrophysical applications, see, e.g.,

R.A. Konoplya, A. Zhidenko, Rev.M.Phys. (2011); arXiv:1102.4014,

and maybe also in cosmology,

M. Maggiore et al, arXiv:1403.6068; arXiv:1402.0448.

General curved background.

Using normal coordinates

F. Salles and I.Sh., Phys. Rev. (2014).

$$H^{\mu\nu,\,\alpha\beta}\,\bar{h}^{\perp}_{\alpha\beta} \,=\, 0\,,$$

and, in the first order in curvature,

$$H_{\mu\nu,\,\alpha\beta} \,=\, -\,\frac{a_1}{2}\,\delta_{\mu\nu,\,\alpha\beta}\Box^2 + D^{\rho\lambda}{}_{\mu\nu,\alpha\beta}\nabla_\rho\nabla_\lambda \,+\, W_{\mu\nu,\,\alpha\beta}\,,\,,$$

$$D^{\rho\lambda}{}_{\mu\nu,\alpha\beta} \,=\, 2a_1 g_{\nu\beta}R_{\alpha\,\cdot\,\cdot\,\mu}^{\;\;\rho\lambda} + a_1 g^{\rho\lambda}\big(2g_{\nu\beta}R_{\alpha\mu} - R_{\mu\alpha\nu\beta}\big)$$

$$+\,\Big(\frac{M_P^2}{64\pi} - \frac{a_1}{6}R - \frac{a_4}{2}R\Big)g^{\rho\lambda}\delta_{\mu\nu,\alpha\beta}\,;$$

$$W_{\mu\nu,\,\alpha\beta} \,=\, \frac{M_P^2}{64\pi}\,\big(R_{\mu\alpha\nu\beta} + 3\,R_{\mu\alpha}\,g_{\nu\beta} - R\,\delta_{\mu\nu,\alpha\beta}\big)\,.$$

The use of normal coordinates can be useful in providing the framework for the general study of stability

$$g_{\alpha\beta}(y) = \eta_{\alpha\beta} - \frac{1}{3}\overset{\circ}{R}_{\mu\alpha\nu\beta}\,y^\mu y^\nu + \dots .$$

$$\Box h^{\alpha\beta} = (\Box h^{\alpha\beta})^{(0)} + (\Box h^{\alpha\beta})^{(1)} + (\Box h^{\alpha\beta})^{(2)} + \,..,$$

$$h_{\mu\nu}(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3}\,h_{\mu\nu}(\vec{k},t)\,e^{i\vec{k}\cdot\vec{r}} .$$

$$H_{\mu\nu,\,\alpha\beta} = -\frac{a_1}{2}\,\delta_{\mu\nu,\,\alpha\beta}(\Box^2 h^{\alpha\beta})^{(0)} + 2a_1\eta_{\nu\beta}\,\overset{\circ}{R}{}_{\alpha\,\cdot\,\cdot\,\mu}^{\;\;\rho\lambda}\,\partial_\rho\partial_\lambda$$

$$+\Big[\Big(\frac{a_1}{6}\overset{\circ}{R} + \frac{a_4}{2}\overset{\circ}{R} + \frac{M_P^2}{64\pi}\Big)\delta_{\mu\nu,\,\alpha\beta} + 2a_1\,\eta_{\nu\beta}\,\overset{\circ}{R}_{\alpha\mu} - a_1\overset{\circ}{R}_{\mu\alpha\nu\beta}\Big]\Box$$

$$+\frac{M_P^2}{64\pi}\,\big(\overset{\circ}{R}_{\mu\alpha\nu\beta} + 3\,\eta_{\mu\alpha}\,\overset{\circ}{R}_{\nu\beta} - \overset{\circ}{R}\,\delta_{\mu\nu,\,\alpha\beta}\big) .$$

Conclusions

- One should definitely quantize both matter and gravity, for otherwise the QG theory would not be complete. And it can not be done without HD terms.

- For QG with higher derivatives (HDQG) the propagator includes massive nonphysical mode(s) called ghosts.

- These massive ghosts are capable to produce terrible instabilities, but ... for this end there should be at least one such ghost excitation in the initial spectrum.

- At least in the cosmological case, the ghost is not actually generated below Planck scale.

- The final conclusion is that the HDQG may be a perfect candidate to be an effective QG below the Planck scale.