New geometries for the characterization of dark matter phenomena

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- The standard characterization of dark matter phenomena is through models that assume the generally accepted cold dark matter model. However, when studying dark matter phenomena with different techniques one often finds non-trivial disagreement among the measurements.
- Notably, when estimating the matter content in a region using gravitational weak lensing effects and dynamical studies, the different techniques do not coincide in the estimated value.
- These problems might be related to the way one normally deals with inhomogeneities in cosmology. We will comment briefly on the inherent problems involved in the notion of averaging of tensors; that contribute to unexpected terms in the energy momentum tensor.
- In a previous study of weak lensing we have noticed that a spacelike contribution of the energy-momentum tensor has been neglected in previous works.
- We present some new geometries that involve new kind of energy momentum tensors which are suitable for the description of dark matter phenomena.

What could be missing from the standard picture?

The problem with implicit averages

- In a simple cosmological model one can consider a Universe made out of small pieces of matter distributed in corresponding islands. If a photon would reach us from one of those bodies it would feel:

  a vanishing Ricci tensor, $R_{ab} = 0$ and a non-vanishing Weyl tensor, ${W_{abc}}^{d} \neq 0$.
- While in a smooth averaged description, one would have the contrary, namely:

  a non-vanishing Ricci tensor, $R_{ab} \neq 0$ and a vanishing Weyl tensor, ${W_{abc}}^{d} = 0$.

  As is the case in the Robertson-Walker spacetimes.

The standard approach to weak lenses

- In standard textbooks, as *Gravitational lenses*, P. Schneider, J. Ehlers and E.E. Falco (1992), one finds that the deflection angle is expressed by:

$$\hat{\alpha}(\vec{\xi}) = \sum_i \frac{4Gm_i}{c^2} \frac{\vec{\xi} - \vec{\xi}_i}{|\vec{\xi} - \vec{\xi}_i|^2}, \tag{1}$$

where $\vec{\xi}$ describes de position of the light ray in the lens plane, and $\vec{\xi}_i$ that of the mass m_i; which movements are assumed to be negligible.

Then, expressing this equation in terms of the surface mass density $\Sigma(\vec{\xi})$, for a continuum distribution, one obtains

$$\hat{\alpha}(\vec{\xi}) = \frac{4G}{c^2} \int_{\mathbb{R}^2} \Sigma(\vec{\xi}') \frac{\vec{\xi} - \vec{\xi}_i}{|\vec{\xi} - \vec{\xi}_i|^2} d^2\xi', \tag{2}$$

where $\Sigma(\vec{\xi})$ is the mass density projected onto a plane perpendicular to the light path.

- Instead we have shown in Phys. Rev., D83:083007, (2011), Emanuel Gallo and O.M.M., the following expressions for the bending angle in terms of energy-momentum components and the mass content $M(r)$, of a spherically symmetric stationary spacetime

$$\boxed{\alpha(J) = J \int_{-d_l}^{d_{ls}} \left[\frac{3J^2}{r^2} \left(\frac{M(r)}{r^3} - \frac{4\pi}{3} \varrho(r) \right) + 4\pi \left(\varrho(r) + P_r(r) \right) \right] dy} ; \qquad (3)$$

where $J = |\vec{\xi}|$ is the impact parameter and $r = \sqrt{J^2 + y^2}$.
Let us observe the appearance of a term proportional to the radial component of the energy-momentum tensor; namely P_r.

This suggested us to consider a simple model with $P_r \neq 0$ and $M(r) = 0$, $\rho(r) = 0$

A stationary spherically symmetric spacetime can be expressed in terms of the standard line element

$$ds^2 = a(r)\, dt^2 - b(r)\, dr^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2); \qquad (4)$$

where $M(r)$ is defined from

$$b(r) = \frac{1}{1 - \frac{2M(r)}{r}}; \tag{5}$$

which in our case is zero.
The corresponding energy-momentum tensor of this solution[GM12] has components:

$$T_{tt} = 0\,; \tag{6}$$
$$T_{rr} = P_r = \frac{1}{4\pi r^2 \ln(\frac{r}{\mu})}\,; \tag{7}$$
$$T_{\theta\theta} = 0; \tag{8}$$
$$T_{\varphi\varphi} = 0\,. \tag{9}$$

So, we see that:

The geometry has a curvature logarithmic singularity at the internal radius $r = \mu$.

Introduction: What could be missing from the standard picture? V

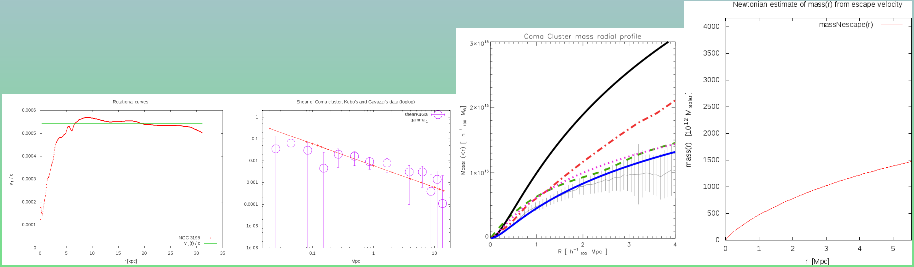

Figure: Rotational curve for galaxy NGC 3198 (red), and for this geometry (green). Fitting the geometry to adjust the observations of [JAJ$^+$07, G$^+$09] for the shear of Coma cluster. Graph from Serra-Dominguez[SR11] article. Estimate of the matter content based on the scape velocity concept. (Although the solution has no mass)

This geometry describes fairly well dark matter phenomena.(rotation curves, weak lens, scape velocities)

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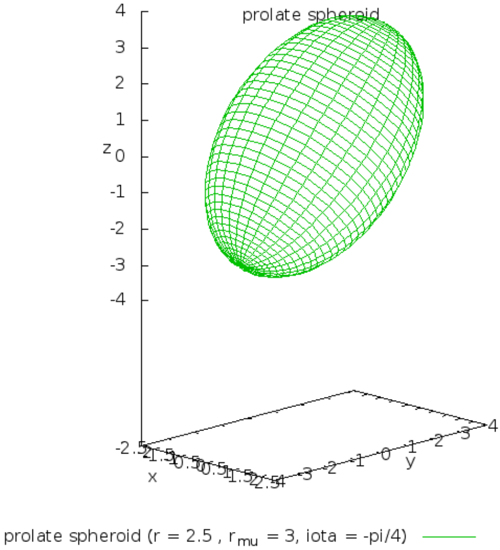
prolate spheroid
z
4
3
2
1
0
-1
-2
-3
-4
x
y
prolate spheroid (r = 2.5 , r mu = 3, iota = -pi/4)

Using the hyperbolic coordinate

The metric

We will consider spacetimes with spheroidal symmetry of the form

$$\begin{aligned} ds^2 =& a(\xi,t)dt^2 - b(\xi,t)r_\mu^2(\sinh^2(\xi)+\sin^2(\theta))d\xi^2 \\ & - r_\mu^2\left((\sinh^2(\xi)+\sin^2(\theta))d\theta^2 \ +\sinh^2(\xi)\sin^2(\theta)d\phi^2\right), \end{aligned} \tag{10}$$

In particular we present the static solution given by

$$a = a_0(\xi + C)^2, \tag{11}$$

and

$$b = 1. \tag{12}$$

The Einstein tensor

The corresponding components of the Einstein tensor which are different from zero are:

$$G_{\xi\xi} = -\frac{(2\cosh^2(\xi) - 2 + \sin^2(\theta))\cosh(\xi)\sinh(\xi)}{(\cosh^2(\xi) + \sin^2(\theta) - 1)(\cosh^2(\xi) - 1)(\xi + C)}, \tag{13}$$

$$G_{\xi\theta} = -\frac{\cos(\theta)\sin(\theta)}{(\cosh^2(\xi) - 1 + \sin^2(\theta))(\xi + C)}, \tag{14}$$

$$G_{\theta\theta} = -\frac{\cosh(\xi)\sin^2(\theta)\sinh(\xi)}{(\cosh^2(\xi) - 1 + \sin^2(\theta))(\cosh^2(\xi) - 1)(\xi + C)}. \tag{15}$$

Using the radial coordinate

The metric

From the relation

$$\xi = \text{arcsinh}(\frac{r}{r_\mu}) = \ln\left(\frac{r}{r_\mu} + \sqrt{(\frac{r}{r_\mu})^2 + 1}\right); \tag{16}$$

or alternatively

$$r = r_\mu \sinh(\xi); \tag{17}$$

one can express the metric as:

$$ds^2 = a(r)dt^2 - \left((r^2 + r_\mu^2 \sin^2(\theta))\left(\frac{dr^2}{r^2 + r_\mu^2} + d\theta^2\right) + r^2 \sin^2(\theta) d\phi^2\right), \tag{18}$$

and the timelike component of the metric is

$$a = a_0 \left(\ln\left(\frac{r}{r_\mu} + \sqrt{(\frac{r}{r_\mu})^2 + 1}\right) + C\right)^2. \tag{19}$$

Using the radial coordinate II

The Einstein tensor

The corresponding components of the Einstein tensor which are different from zero are:

$$G_{rr} = -\frac{(2r^2 + r_\mu^2 \sin^2(\theta))}{\sqrt{r^2 + r_\mu^2}\left(\log\left(\frac{\sqrt{r^2+r_\mu^2}+r}{r_\mu}\right) + C\right)(r_\mu^2 \sin^2(\theta) + r^2)r}, \tag{20}$$

$$G_{r\theta} = -\frac{r_\mu^2 \cos(\theta)\sin(\theta)}{\sqrt{r^2 + r_\mu^2}\left(\log\left(\frac{\sqrt{r^2+r_\mu^2}+r}{r_\mu}\right) + C\right)(r_\mu^2 \sin^2(\theta) + r^2)}, \tag{21}$$

$$G_{\theta\theta} = -\frac{(r^2 + r_\mu^2) r_\mu^2 \sin^2(\theta)}{\sqrt{r^2 + r_\mu^2}\left(\log\left(\frac{\sqrt{r^2+r_\mu^2}+r}{r_\mu}\right) + C\right)(r_\mu^2 \sin^2(\theta) + r^2)r}. \tag{22}$$

The Riemann tensor

The components of the Riemann tensor which are different from zero are:

$$R_{trtr} = \frac{-(\log(\frac{\sqrt{r^2+r_\mu^2}+r}{r_\mu}) + C)a_0 r}{\sqrt{r^2+r_\mu^2}(\sin(\theta)^2 r_\mu^2 + r^2)}, \tag{23}$$

$$R_{trt\theta} = \frac{-(\log(\frac{\sqrt{r^2+r_\mu^2}+r}{r_\mu}) + C)\cos(\theta)\sin(\theta)a_0 r_\mu^2}{\sqrt{r^2+r_\mu^2}(\sin(\theta)^2 r_\mu^2 + r^2)}, \tag{24}$$

$$R_{t\theta t\theta} = \frac{(\log(\frac{\sqrt{r^2+r_\mu^2}+r}{r_\mu}) + C)(r^2+r_\mu^2)a_0 r}{\sqrt{r^2+r_\mu^2}(\sin(\theta)^2 r_\mu^2 + r^2)}, \tag{25}$$

$$R_{t\phi t\phi} = \frac{(\log(\frac{\sqrt{r^2+r_\mu^2}+r}{r_\mu}) + C)(r^2+r_\mu^2)\sin(\theta)^2 a_0 r}{\sqrt{r^2+r_\mu^2}(\sin(\theta)^2 r_\mu^2 + r^2)}. \tag{26}$$

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The adapted coordinate system I

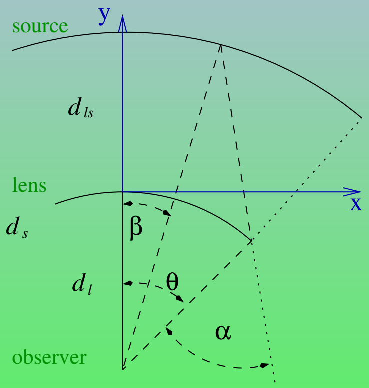


Figure: Standard notation for deviation angles and background coordinate system. d_s denotes the distance to the source of the image; d_l to the lens and d_{ls} the lens-source distance.

We use coordinates (x, z) for the plane of the lens.

A rotated spheroid

Relation among the coordinates

Let now consider two sets of Cartesian coordinates (x, y, z) and (x', y', z'). The first is adapted to the discussion of a gravitational lens, as in our previous article; so that the plane $y = 0$ is identified with the plane of the lens. The observer is at the position $y = -d_l$; and the source at the plane $y = d_s$. The primed coordinate system is adapted to the spheroidal distribution and such that the z' axis has an inclination angle ι with respect to the z axis, as measured with respect to the x axis. So that the relation between the two coordinate systems is given by:

$$x' = x, \tag{27}$$

$$y' = y\cos(\iota) + z\sin(\iota), \tag{28}$$

$$z' = z\cos(\iota) - y\sin(\iota); \tag{29}$$

A rotated spheroid II

or equivalently

$$x = r\sin(\theta)\cos(\phi), \tag{30}$$

$$y = r\sin(\theta)\sin(\phi)\cos(\iota) - \sqrt{r^2 + r_\mu^2}\cos(\theta)\sin(\iota), \tag{31}$$

$$z = \sqrt{r^2 + r_\mu^2}\cos(\theta)\cos(\iota) + r\sin(\theta)\sin(\phi)\sin(\iota). \tag{32}$$

The projection into the $y = 0$ plane

Consider the projection of an spheroid of radius r into de $y = 0$ plane.
One can see that this curve coincide with the ellipse given by:

$$\frac{x^2}{b^2} + \frac{z^2}{a^2} = 1; \tag{33}$$

where $b = r$ and

$$a^2 = r^2 + \mathcal{R}_\mu^2; \tag{34}$$

with

$$\mathcal{R}_\mu = r_\mu\cos(\iota). \tag{35}$$

That is, the projection of an spheroid of radius r and focus at r_μ is an ellipse with focus at $\mathcal{R}_\mu = r_\mu\cos(\iota)$.

Gravitational lens geometry for prolate spheroidal distributions

In the calculation of gravitational lens, one needs to calculate the spinor components of the Ricci tensor Φ_{00} and the Weyl component Ψ_0, with respect to a null tetrad adapted to the null geodesic congruence of the photons.
We choose the null tetrad in the flat background as in our previous article; so that in the (t, x, y, z) frame, one has

$$l^{\underline{a}} = (-1, 0, 1, 0), \tag{36}$$

$$m^{\underline{a}} = \frac{1}{\sqrt{2}}(0, i, 0, 1), \tag{37}$$

$$\bar{m}^{\underline{a}} = \frac{1}{\sqrt{2}}(0, -i, 0, 1), \tag{38}$$

$$n^{\underline{a}} = \frac{1}{2}(-1, 0, -1, 0). \tag{39}$$

Gravitational lens geometry for prolate spheroidal distributions II

To calculate Φ_{00} we make use of (20)-(22) with the relations

$$x' = r\sin(\theta)\cos(\phi), \quad (40)$$

$$y' = r\sin(\theta)\sin(\phi), \quad (41)$$

$$z' = \sqrt{r^2 + r_\mu^2}\cos(\theta); \quad (42)$$

or equivalently

$$r = \sqrt{\frac{x'^2+y'^2+z'^2-r_\mu^2}{2} + \frac{\sqrt{(x'^2+y'^2+z'^2-r_\mu^2)^2+4(x'^2+y'^2)r_\mu^2}}{2}}, \quad (43)$$

$$\theta = \arctan\left(\frac{\sqrt{x'^2+y'^2}\sqrt{r^2+r_\mu^2}}{rz'}\right) \quad (44)$$

$$= \arccos\left(\frac{z'}{\sqrt{r^2+r_\mu^2}}\right),$$

$$\phi = \arctan\left(\frac{y'}{x'}\right); \quad (45)$$

Gravitational lens geometry for prolate spheroidal distributions III

along with the rotation given by (27)-(29).
Let us note that the Ricci component is:

$$\Phi_{00} = -\frac{1}{2} R_{ab} l^a l^b = -\frac{1}{2} G_{ab} l^a l^b; \tag{46}$$

although in our case we will use the approximated null vector given by (36); so that we have

$$\Phi_{00} = -\frac{1}{2} \left(G_{rr} l^r l^r + 2 G_{r\theta} l^r l^\theta + G_{\theta\theta} l^\theta l^\theta \right). \tag{47}$$

It is expected that the $G_{r\theta}$ will give little contribution to the other two.
The Weyl component is given by:

$$\Psi_0 = C_{abcd} l^a m^b l^c m^d; \tag{48}$$

but due to the fact that these vectors are null and orthogonal among them, one also has

$$\Psi_0 = R_{abcd} l^a m^b l^c m^d; \tag{49}$$

which are expressed in terms of (23)-(25).

Gravitational lens geometry for prolate spheroidal distributions IV

We mention this because there are less non-zero Riemann components than Weyl ones; so it is simpler to use the Riemann components.
Also, since the vectors m^a and $\bar{m}^a$ do not have timelike component, one has

$$\Psi_0 = R_{tbtd}(l^t)^2 m^b m^d. \tag{50}$$

We use the fact that $l^t = \frac{1}{\sqrt{g_{tt}}}$, so that

$$\Psi_0 = R_{tbtd}\frac{1}{g_{tt}} m^b m^d. \tag{51}$$

For the calculation of the tetrad components one needs the partial derivatives of the spheroidal coordinates in terms of the Cartesian ones. Let us note that r^2 can be expressed as

$$\begin{aligned} r^2 =& \frac{1}{2}\left(x'^2 + y'^2 + z'^2 - r_\mu^2 + \sqrt{\varpi}\right) \\ =& \frac{1}{2}\left(x^2 + y^2 + z^2 - r_\mu^2 + \sqrt{\varpi}\right); \end{aligned} \tag{52}$$

Gravitational lens geometry for prolate spheroidal distributions V

where

$$\varpi = (x^2 + y^2 + z^2 + r_\mu^2)^2 - 4(z\cos(\iota) - y\sin(\iota))^2 r_\mu^2. \tag{53}$$

Then, one has:

$$\frac{\partial r}{\partial x} = \frac{x}{r}\left(\frac{r^2 + r_\mu^2}{2r^2 - (x^2 + y^2 + z^2) + r_\mu^2}\right), \tag{54}$$

$$\begin{aligned}\frac{\partial r}{\partial y} =& \frac{y}{r}\left(\frac{r^2 + r_\mu^2}{2r^2 - (x^2 + y^2 + z^2) + r_\mu^2}\right)\\ &+ \frac{1}{r}\left(\frac{\sin(\iota)(z\cos(\iota) - y\sin(\iota))r_\mu^2}{\sqrt{\varpi}}\right),\end{aligned} \tag{55}$$

$$\begin{aligned}\frac{\partial r}{\partial z} =& \frac{z}{r}\left(\frac{r^2 + r_\mu^2}{2r^2 - (x^2 + y^2 + z^2) + r_\mu^2}\right)\\ &- \frac{1}{r}\left(\frac{\cos(\iota)(z\cos(\iota) - y\sin(\iota))r_\mu^2}{\sqrt{\varpi}}\right).\end{aligned} \tag{56}$$

Gravitational lens geometry for prolate spheroidal distributions VI

For the angular coordinate θ one has:

$$\frac{\partial\theta}{\partial x} = \frac{z' r}{(r^2 + r_\mu^2)\sqrt{r^2 + r_\mu^2 - z'^2}} \frac{\partial r}{\partial x}, \tag{57}$$

$$\begin{aligned} \frac{\partial\theta}{\partial y} =& \frac{1}{\sqrt{r^2 + r_\mu^2 - z'^2}} \sin(\iota) \\ &+ \frac{z' r}{(r^2 + r_\mu^2)\sqrt{r^2 + r_\mu^2 - z'^2}} \frac{\partial r}{\partial y}, \end{aligned} \tag{58}$$

$$\begin{aligned} \frac{\partial\theta}{\partial z} =& -\frac{1}{\sqrt{r^2 + r_\mu^2 - z'^2}} \cos(\iota) \\ &+ \frac{z' r}{(r^2 + r_\mu^2)\sqrt{r^2 + r_\mu^2 - z'^2}} \frac{\partial r}{\partial z}. \end{aligned} \tag{59}$$

Gravitational lens geometry for prolate spheroidal distributions VII

For the angular coordinate ϕ one needs to calculate its x and z derivatives; which are needed in the calculation of Ψ_0. These are:

$$\frac{d\phi}{dx} = -\frac{y\cos(\iota) + z\sin(\iota)}{x^2 + (y\cos(\iota) + z\sin(\iota))^2}, \tag{60}$$

$$\frac{d\phi}{dz} = \frac{d\arctan(\frac{y'}{x'})}{dz} = \frac{x\sin(\iota)}{x^2 + (y\cos(\iota) + z\sin(\iota))^2}. \tag{61}$$

Gravitational lens geometry for prolate spheroidal distributions VIII

Then, the vector l, in terms of the Cartesian system is given by:

$$l = -\frac{1}{\sqrt{g_{tt}}}\frac{\partial}{\partial t} + \frac{\partial}{\partial y}. \tag{62}$$

In terms of the (t, r, θ, ϕ) coordinate system, it is expressed by:

$$l = -\frac{1}{\sqrt{g_{tt}}}\frac{\partial}{\partial t} + \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}. \tag{63}$$

Similarly, the vector m, in this coordinate system is:

$$\begin{aligned} m &= \frac{1}{\sqrt{2}}\left(i\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right) \\ &= \frac{1}{\sqrt{2}}\left((i\frac{\partial r}{\partial x} + \frac{\partial r}{\partial z})\frac{\partial}{\partial r} + (i\frac{\partial \theta}{\partial x} + \frac{\partial \theta}{\partial z})\frac{\partial}{\partial \theta} + (i\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z})\frac{\partial}{\partial \phi}\right). \end{aligned} \tag{64}$$

Gravitational lens geometry for prolate spheroidal distributions IX

The components we need, then are:

$$l^r = \frac{\partial r}{\partial y}, \tag{65}$$

$$l^\theta = \frac{\partial \theta}{\partial y}. \tag{66}$$

and

$$m^r = \frac{1}{\sqrt{2}}(i\frac{\partial r}{\partial x} + \frac{\partial r}{\partial z}), \tag{67}$$

$$m^\theta = \frac{1}{\sqrt{2}}(i\frac{\partial \theta}{\partial x} + \frac{\partial \theta}{\partial z}), \tag{68}$$

$$m^\phi = \frac{1}{\sqrt{2}}(i\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z}). \tag{69}$$

Gravitational lens geometry for prolate spheroidal distributions X

We finally arrive then at the values

$$\Phi_{00} = -\frac{1}{2}\left(G_{rr} l^r l^r + 2G_{r\theta} l^r l^\theta + G_{\theta\theta} l^\theta l^\theta\right)$$
$$= \frac{1}{2\sqrt{r^2 + r_\mu^2}\left(\log\left(\frac{\sqrt{r^2+r_\mu^2}+r}{r_\mu}\right) + C\right)(r_\mu^2 \sin^2(\theta) + r^2) r}$$
$$\left((2r^2 + r_\mu^2 \sin^2(\theta)) l^r l^r + 2\, r\, r_\mu^2 \cos(\theta) \sin(\theta) l^r l^\theta\right.$$
$$\left. + (r^2 + r_\mu^2) r_\mu^2 \sin^2(\theta) l^\theta l^\theta\right); \quad (70)$$

Gravitational lens geometry for prolate spheroidal distributions XI

while the Weyl component is given by:

$$\begin{aligned}
\Psi_0 =& \frac{1}{g_{tt}} R_{tbtd}\, m^b\, m^d \\
=& \frac{1}{g_{tt}} R_{trtr} (m^r)^2 + \frac{2}{g_{tt}} R_{trt\theta}\, m^r\, m^\theta \\
& + \frac{1}{g_{tt}} R_{t\theta t\theta} (m^\theta)^2 + \frac{1}{g_{tt}} R_{t\phi t\phi} (m^\phi)^2 \\
=& \frac{1}{(\log(\frac{\sqrt{r^2+r_\mu^2}+r}{r_\mu}) + C)\sqrt{r^2 + r_\mu^2}(\sin(\theta)^2 r_\mu^2 + r^2)} \\
& \Big(- r(m^r)^2 - 2\cos(\theta)\sin(\theta)\, r_\mu^2\, m^r\, m^\theta \\
& + (r^2 + r_\mu^2) r (m^\theta)^2 + (r^2 + r_\mu^2) r \sin(\theta)^2 (m^\phi)^2 \Big).
\end{aligned} \tag{71}$$

The optical scalars I

The optical scalars

Let us recall from [GM11] the the optical scalars, in the thin lens approximation, are given by:

$$\kappa = \frac{d_l d_{ls}}{d_s} \hat{\Phi}_{00}, \tag{72}$$

$$\gamma_1 + i\gamma_2 = \frac{d_l d_{ls}}{d_s} \hat{\Psi}_0, \tag{73}$$

where

$$\begin{aligned} \hat{\Phi}_{00} &= \int_0^{d_s} \Phi_{00} d\lambda, \\ \hat{\Psi}_0 &= \int_0^{d_s} \Psi_0 d\lambda, \end{aligned} \tag{74}$$

are the projected curvature scalars along the line of sight.

Content

For the numerical calculation we have taken the following values:

- The parameter C was take as $-\ln(\mu)$, from reference [GM12] which it was adjusted to the observations of weak lens in the Coma cluster.
- The radius r_μ was arbitrarily taken to have the value 3Mpc.
- The rotation angle ι was chosen to be $\frac{\pi}{4}$.
- The lens distances were taken as: $d_l = 97.10$Mpc, $d_s = 1068.03$Mpc, $d_{ls} = 970.92$Mpc; which are values from the Coma cluster used in our previous work.
- The integration was carried out using Chebyshev-Gauss techniques. The number of points evaluated was automatically adjusted to a chosen tolerance in the result.

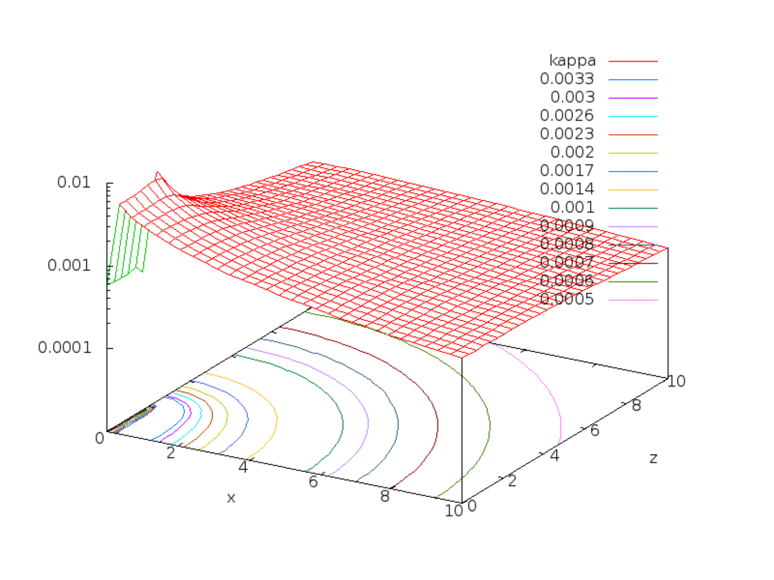


Figure: The expansion κ plotted in a log scale. One can see that it copies the geometry of the projected spheroids.

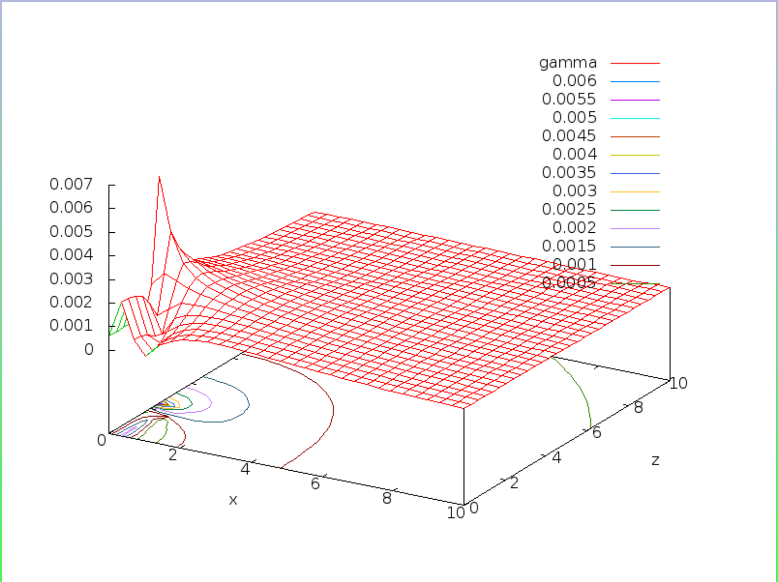
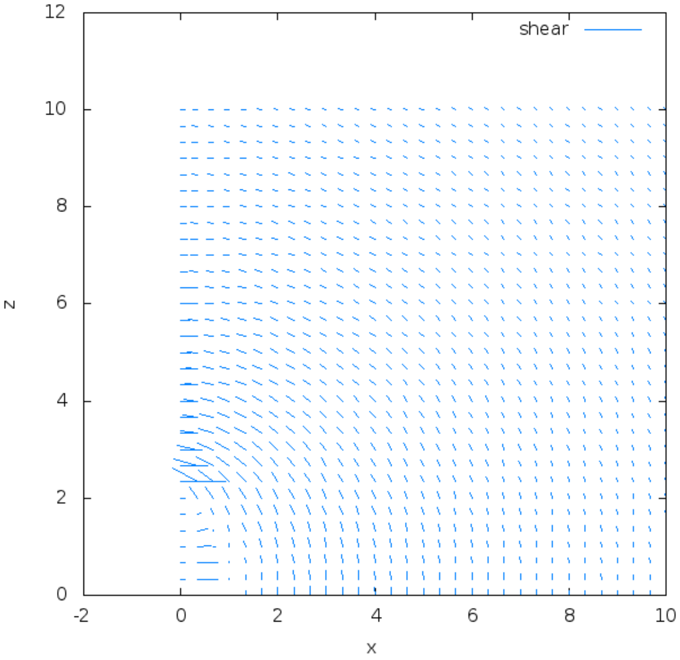


Figure: The modulus γ of the shear expansion. The contour curves are more complicated in this case.



shear
12
10
8
6
4
2
0
z
-2
0
2
4
6
8
10
x

Remark on the axially symmetric case I

The axially symmetric case

We have seen in our previous work [GM11] that in the axially symmetric case, the spin 2 nature of the Weyl component Ψ_0 is such that its phase can be extracted from the integration, along the path of the photon, so that:

$$\hat{\Psi}_0(J) \quad = \quad -e^{2i\vartheta}\hat{\psi}_0(J); \tag{75}$$

where $\hat{\psi}_0(J)$ is a real quantity. For this reason, the shear can be expressed by

$$\gamma_1 + i\gamma_2 = -\gamma e^{2i\vartheta}; \tag{76}$$

with

$$\gamma = \frac{d_{ls} d_l}{d_s}\hat{\psi}_0(J). \tag{77}$$

This property can not be translated to the spheroidal case; which makes the phase dependence to be much more complicated. In particular, this seems to be the reason why the modulus of the shear does not copy the geometry of the projected spheroids.

Content

Spacetimes with prolate spheroidal symmetry and mass I

The zero mass spacetime just presented can be generalized to spacetimes with mass content; as we do next.

The metric

Here we present a new solution with mass content, spheroidal symmetry and a non-trivial spacelike component of the energy momentum tensor. The metric is:

$$ds^2 = a(r)dt^2 - \left((r^2 + r_\mu^2 \sin^2(\theta)) \left(\frac{dr^2}{r^2 - 2M(r)r + r_\mu^2} + d\theta^2 \right) + r^2 \sin^2(\theta) d\phi^2 \right), \quad (78)$$

and the timelike component of the metric is

$$a = a_0 \left(\ln \left(\frac{r}{r_\mu} + \sqrt{(\frac{r}{r_\mu})^2 + 1} \right) + C \right)^2, \quad (79)$$

and where $M(r)$ is:

$$M(r) = \frac{M_*}{r_*} r \quad \text{for} \quad r \leqslant r_* \text{ and } M(r) = M_* \quad \text{for} \quad r > r_* \text{ (isothermal) or} \quad (80)$$

$$M(r) = 4\pi \rho_* r_*^3 \left(\log(1 + \frac{r}{r_*}) - \frac{\frac{r}{r_*}}{1 + \frac{r}{r_*}} \right) \quad \text{(NFW).} \quad (81)$$

Spacetimes with prolate spheroidal symmetry and mass II

The Einstein tensor

The corresponding timelike component of the Einstein tensor is:

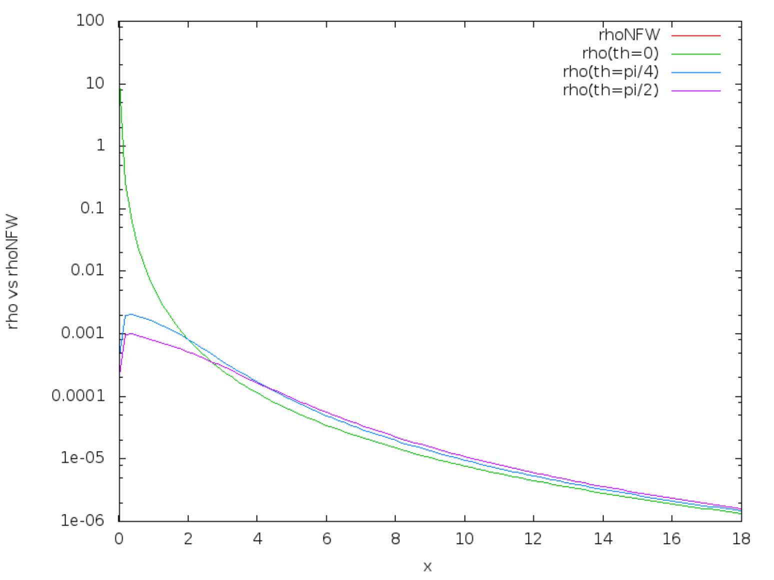
$$G_{tt} = -\left((5r^2 + r_\mu^2 \sin^2(\theta))Mr_\mu^2 + (\sin(\theta)^2 r_\mu^2 + 2r^2)(\sin(\theta)^2 r_\mu^2 + r^2)\frac{dM}{dr}r\right) \frac{(\log((\sqrt{(}r^2 + r_\mu^2) + r)/r_\mu) + C)^2 a_0}{(r_\mu^2 \sin^2(\theta) + r^2)^3 r}; \tag{82}$$

so that the natural definition of mass would give a non-zero answer.

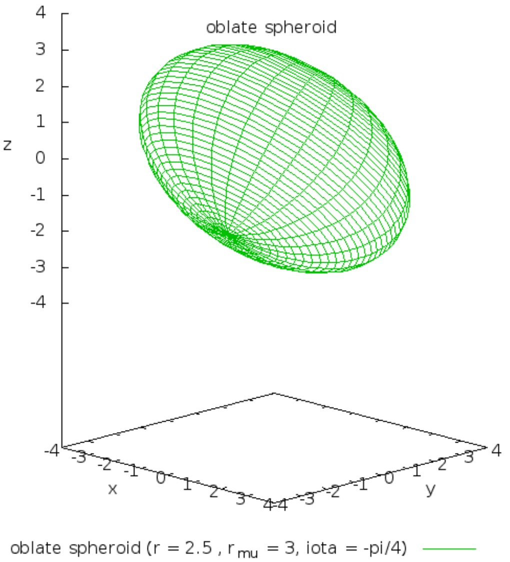
The Riemann tensor

The components of the Riemann tensor which are different from zero are R_{trtr}, $R_{trt\theta}$, $R_{t\theta t\theta}$, $R_{t\phi t\phi}$ $R_{r\theta r\theta}$, $R_{r\phi r\phi}$ and $R_{\theta\phi\theta\phi}$.

Spacetimes with prolate spheroidal symmetry and mass III



Content


oblate spheroid
z
4
3
2
1
0
-1
-2
-3
-4
x
-4 -3 -2 -1 0 1 2 3 4
y
-4 -3 -2 -1 0 1 2 3 4
oblate spheroid (r = 2.5 , r mu = 3, iota = -pi/4)

Using the hyperbolic coordinate

The metric

We will consider spacetimes with oblate spheroidal symmetry of the form

$$\begin{aligned} ds^2 =& a(\xi,t)dt^2 - b(\xi,t)r_\mu^2(\cosh^2(\xi) - \sin^2(\theta))d\xi^2 \\ & - r_\mu^2\left((\cosh^2(\xi) - \sin^2(\theta))d\theta^2 + \cosh^2(\xi)\sin^2(\theta)d\phi^2\right), \end{aligned} \tag{83}$$

In particular we present the static solution given by

$$a = a_0(\xi + C)^2, \tag{84}$$

and

$$b = 1. \tag{85}$$

The Einstein tensor

The corresponding components of the Einstein tensor which are different from zero are:

$$G_{\xi,\xi} = -\frac{(2\cosh^2(\xi) - \sin^2(\theta))\sinh(\xi)}{(\cosh^2(\xi) - \sin^2(\theta))\cosh(\xi)(\xi + C)}, \quad (86)$$

$$G_{\xi,\theta} = \frac{\cos(\theta)\sin(\theta)}{(\cosh^2(\xi) - \sin^2(\theta))(\xi + C)}, \quad (87)$$

$$G_{\theta,\theta} = -\frac{\sin^2(\theta)\sinh(\xi)}{(\cosh^2(\xi) - \sin^2(\theta))\cosh(\xi)(\xi + C)}. \quad (88)$$

Using the radial coordinate

The metric

From the relation

$$r = r_\mu \sinh(\xi); \tag{89}$$

one can express the metric as:

$$ds^2 = a(r)dt^2 - \left((r^2 + r_\mu^2 \cos^2(\theta)) \left(\frac{dr^2}{r^2 + r_\mu^2} + d\theta^2 \right) + (r^2 + r_\mu^2) \sin^2(\theta) d\phi^2 \right), \tag{90}$$

and the timelike component of the metric is

$$a = a_0 \left(\ln \left(\frac{r}{r_\mu} + \sqrt{(\frac{r}{r_\mu})^2 + 1} \right) + C \right)^2. \tag{91}$$

From the definition of prolate spheroidal coordinates one can see that the radial coordinate r is related to the Cartesian coordinates by:

$$r^2 = \frac{1}{2} \left(x'^2 + y'^2 + z'^2 - r_\mu^2 + \sqrt{(x'^2 + y'^2 + z'^2 - r_\mu^2)^2 + 4z'^2 r_\mu^2} \right). \tag{92}$$

Content

The zero mass spacetime just presented can be generalized to spacetimes with mass content; as we do next.

The metric

Here we present a new solution with mass content, spheroidal symmetry and a non-trivial spacelike component of the energy momentum tensor. The metric is:

$$ds^2 = a(r)dt^2 - (r^2 + r_\mu^2 \cos^2(\theta)) \frac{dr^2}{r^2 - 2M(r)r + r_\mu^2} - \Big((r^2 + r_\mu^2 \cos^2(\theta)) d\theta^2 + (r^2 + r_\mu^2) \sin^2(\theta) d\phi^2 \Big), \quad (93)$$

and the timelike component of the metric is

$$a = a_0 \left(\ln \left(\frac{r}{r_\mu} + \sqrt{(\frac{r}{r_\mu})^2 + 1} \right) + C \right)^2, \quad (94)$$

Oblate spheroidal geometries with mass II

and where $M(r)$ is:

$$M(r) = \frac{M_*}{r_*} r \quad \text{for} \quad r \leqslant r_* \text{ and } M(r) = M_* \quad \text{for} \quad r > r_* \text{ (isothermal) or} \tag{95}$$

$$M(r) = 4\pi \rho_* r_*^3 \left(\log(1 + \frac{r}{r_*}) - \frac{\frac{r}{r_*}}{1 + \frac{r}{r_*}} \right) \qquad \text{(NFW).} \tag{96}$$

These mass distributions are the natural generalization of the isothermal mass density to the spheroidal geometry and of the NFW distribution to spheroidal geometry.

Content

Final comments I

- We have presented several new static exact solution of the Einstein equations, with spheroidal symmetry. Some of them have $T_{tt} = 0$, and therefore they haves zero mass, although with a non-trivial gravitational field.
- It is the natural generalization of a previous geometry we presented before[GM12], with spherical symmetry; that adequately represents dark mater phenomena.
- The behaviour of the shear in the weak lens calculation is not yet well understood; but it might indicate a non-trivial behaviour of the spin 2 nature of the Weyl Ψ_0 component.
- We can naturally generalize these solutions to other matter distributions with spheroidal symmetry.

⊛

We wish to develop these techniques for applications to typical non-spheric systems as binary systems, irregular clusters, galaxies, etc.

Final comments II

Thank you!

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