Hamiltonian formulation of modified teleparallel theories of gravity

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Outline

- Introduction to teleparallel gravity
- Modified teleparallel gravity
- Counting degrees of freedom
- An overview of constrained Hamiltonian systems
- Hamiltonian formulation of TEGR and f(T)
- Concluding remarks

Introduction to teleparallel gravity

General Relativity

The general theory of relativity is a metric theory of gravitation. Its equations of motion describe the relation between the geometry of spacetime and the energy-momentum contained on it.
It can be formulated from first principles by considering the action

$$S_{GR}(g^{\mu\nu}) = \int d^4x\sqrt{-g}\,g^{\mu\nu}R_{\mu\nu} = \int d^4x\sqrt{-g}\,R$$

in metric formalism. Varying with respect to the metric gives us the Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

where $T_{\mu\nu}$ is the energy-momentum tensor and describes the energy content of the system in consideration.

Tetrad formalism

General Relativity can be reformulated in a teleparallel framework by taking the field of tetrads as the dynamical variable instead of the metric tensor. The tetrad is a basis $e_a(x)$, $a = 0, 1, 2, 3$, of vectors in the spacetime. Each vector e_a can be decomposed in a coordinate basis with components e_a^{μ}, such that the orthonormality condition reads

$$\eta_{ab} = g_{\mu\nu} e_a^{\mu} e_b^{\nu},$$

where $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$. With the help of the components e^b_{μ} of the co-frame e^a that satisfies $e_a^{\mu} e^b_{\mu} = \delta_a^b$ it can be obtained the metric starting from the tetrad

$$g_{\mu\nu} = \eta_{ab} e^a_{\mu} e^b_{\nu} \to \sqrt{-g} = \det[e^a_{\mu}] = e.$$

Weitzenböck connection and torsion

The teleparallel equivalent of general relativity (TEGR) was originally developed by Einstein in an attempt of unifying gravity and electromagnetism. The TEGR Lagrangian does not contain second derivatives because it is quadratic in the tensor

$$T^{\mu}{}_{\nu\rho} = e^{\mu}_{a}(\partial_{\nu} e^{a}{}_{\rho} - \partial_{\rho} e^{a}{}_{\nu})$$

which resembles the electromagnetic field tensor. This can be regarded as the torsion of the Weitzenböck connection,

$$\Gamma^{\mu}{}_{\rho\nu} = e^{\mu}_{a}\partial_{\nu} e^{a}{}_{\rho} = -e^{a}{}_{\rho}\partial_{\nu} e^{\mu}_{a}$$

The Weitzenböck connection differs from Levi-Civita connection in a tensor $K^{\mu}{}_{\rho\nu}$ named contorsion. Also, it vanishes the curvature tensor, so this connection has torsion but not curvature.

A. Einstein, Sitzungsber. Preuss. Akad. Wiss. (1928), 217; ibid., (1939), 401; A. Einstein, Math. Annal. 102 (1930), 685.

Teleparallel gravity lagrangian

The TEGR Lagrangian is given by

$$S_T[e^a] = \frac{1}{2\kappa}\int d^4x\ e\ S_\rho{}^{\mu\nu}\, T^\rho{}_{\mu\nu},$$

where the torsion $T^\rho{}_{\mu\nu}$ and the contorsion $K^{\mu\nu}{}_\rho$ appears; the latter encoded in the tensor $S_\rho{}^{\mu\nu}$ by the expression

$$2\, S_\rho{}^{\mu\nu} \equiv \underbrace{\frac{1}{2}\left(T_\rho{}^{\mu\nu} - T^{\mu\nu}{}_\rho + T^{\nu\mu}{}_\rho\right)}_{contorsion\ K^{\mu\nu}{}_\rho} + T_\lambda{}^{\lambda\mu}\ \delta^\nu_\rho - T_\lambda{}^{\lambda\nu}\ \delta^\mu_\rho\,.$$

It is defined the so-called torsion scalar by

$$\mathcal{T} = S_\rho{}^{\mu\nu}\, T^\rho{}_{\mu\nu}$$

so the action is written as

$$S_T[e^a] = \frac{1}{2\kappa}\int d^4x\ e\mathcal{T}$$

The equations of motion of TEGR are

$$\frac{1}{e}\partial_\mu(e\ S_a{}^{\mu\nu}) + e_a^\lambda(S_\rho{}^{\mu\nu}T^\rho{}_{\mu\lambda} - \frac{1}{4}\delta_\lambda^\rho S_\rho{}^{\mu\nu}T^\rho{}_{\mu\nu}) = 4\pi G\ e_a^\lambda T_\lambda^\rho$$

where T_λ^ρ is the energy-momentum tensor.
The equivalence of this theory and the Einstein-Hilbert action comes from the fact that their Lagrangians differ in a four-divergence:

$$\mathcal{T} = -R - 2\nabla_\rho(T_\mu{}^{\mu\rho})$$

When plugged into the action, the four-divergence is integrated out and the theories are equivalent.

Modified teleparallel gravity

Modified teleparallel gravity

In the same spirit as f(R) gravity, it can be studied f(T) gravity, which consists in the deformation of the teleparallel action, obtaining

$$S_T[e^a] = \frac{1}{2\kappa}\int d^4x\; ef(\mathcal{T}).$$

Unlike f(R) theories, the dynamical equations in f(T) theories are always second order, since the Lagrangian does not contain second derivatives. They are

$$\begin{aligned} 4\left(\frac{1}{e}\partial_\mu(eS_a{}^{\mu\nu}) + e_a^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\mu\nu}\right)f'(\mathcal{T}) + \\ 4S_a{}^{\mu\nu}\partial_\mu(\mathcal{T})f''(\mathcal{T}) - e_a^\nu f(\mathcal{T}) \quad &= \quad -2\kappa e_a^\lambda T_\lambda^\nu, \end{aligned}$$

where T_λ^ν is the energy-momentum tensor.

R. Ferraro and F. Fiorini, Phys. Rev. D 75, 084031 (2007), R. Ferraro and F. Fiorini, Phys. Rev. D 78, 124019 (2008)

These second order equations of motion come together with the loss of local Lorentz invariance. Under a local Lorentz transformation in the tangent space

$$e^b \rightarrow e^{b'} = \Lambda^{b'}_{a}(x) e^a,$$

the transformed tetrad $e^{b'}$ is not a solution of the field equations. Nonetheless, this transformation keep the metric $g_{\mu\nu}$ unchanged.

The four-divergence that differentiates TEGR with GR is not invariant under this transformation. This fact is harmless when $f(T) = \mathcal{T}$, but for a more general f we have

$$f(\mathcal{T}) = f(-R - 2\nabla_\rho T_\mu{}^{\mu\rho}),$$

with the four-divergence not necessarily being integrated out, and therefore not assuring its invariance under $\Lambda^{b'}_{a}(x)$.

This apparent loss of local Lorentz invariance can be seen as by the theory having more degrees of freedom, whose interpretation is an open question.

Counting degres of freedom

Electromagnetism d.o.f.

We can count naively the degrees of freedom of TEGR and GR by looking at the degrees of freedom removed by gauge transformations.

We will look at the electromagnetism as an academic example. The electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is invariant under the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \xi$$

There is **1** generator ξ of gauge transformations
$\Longrightarrow$ implies **2** spurious degrees of freedom (A_0 and the longitudinal oscillations of A_μ)
$\Longrightarrow$ so $4 - 2 = \mathbf{2}$ degrees of freedom remain

General Relativity d.o.f.

The curvature $R_{\mu\nu\lambda\rho}$ associated with the Levi-Civita connection, in the weak field approximation, is invariant under the infinitesimal gauge transformation

$$g_{\mu\nu} \to g_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

There are **4** generators ξ_μ of gauge transformations
$\Longrightarrow$ **8** spurious degrees of freedom
$\Longrightarrow$ therefore $10 - 8 =$ **2** degrees of freedom remain

Teleparallel gravity d.o.f.

Torsion $T^a_{\mu\nu} = e^a_{\nu,\mu} - e^a_{\mu,\nu}$ associated with the Weitzenbock connection is invariant under the transformation

$$e^a_{\ \mu} \rightarrow e^a_{\ \mu} + \partial_\mu \xi^a$$

4 generators ξ^a of gauge transformations
$\Longrightarrow$ **8** spurious degrees of freedom
$\Longrightarrow$ **8** $= 16 - 8$ degrees of freedom remain.
However: local Lorentz invariance kills **6** additional degrees of freedom, as should be expected from the fact that TEGR is equivalent to GR.

An overview of constrained Hamiltonian systems

In Hamiltonian mechanics it is introduced the momenta variable, defined as

$$p_n = \frac{\partial L}{\partial \dot{q}_n} = f_n(q, \dot{q}).$$

In gauge systems not all p_n are independents. It will be a certain number of relations between p's and q's in the form

$$\phi_m(q,p) = 0, \qquad (m = 1, 2, ..., M)$$

These are called primary constraints. A secondary constraint is one that is not primary, i.e. it holds when the equations of motion are satisfied, but need not hold if they are not satisfied. They arise from the condition that the primary constrants should be preserved in time, that is

$$\dot{\phi}_m(q,p) = 0$$

It is said that a relationship $\gamma_\alpha(q,p)=0$ is first class if its Poisson bracket with all the other constraints ϕ_j vanishes weakly, that is

$$\{\gamma_a, \phi_j\} \approx 0$$

We state that a relation $\chi_\alpha(q,p)=0$ is second class if it is not first class.

Primary and secondary constraints can be first or second class, making a subdivision into four categories.

Each physical state will have an unique set of canonical variables (q, p) which satisfy imposed constraints. Since fixing the gauge the set of all present constraints is second class, the number of canonical variables eliminated through eliminating that constraints will be equal to the number of present constraints.
With this in mind, we arrive to the following formula for counting physical degrees of freedom

$$\#\text{d.o.f} = \frac{1}{2}\#(p_n, q_n) - \frac{1}{2}\#\chi_\alpha - \#\gamma_a$$

$$\begin{pmatrix} \#\text{ degrees of} \\ \text{freedom} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \#\text{ canonical} \\ \text{variables} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \#2^{nd}\text{ class} \\ \text{constraints} \end{pmatrix} - \begin{pmatrix} \#1^{st}\text{class} \\ \text{constraints} \end{pmatrix}$$

Hamiltonian formulation of TEGR and f(T) gravity

Hamiltonian formulation of teleparallel gravity

Teleparallel gravity is formulated in a first order differential formulation of the lagrangian density. It is introduced an auxiliary field quantity $\phi_{abc} = -\phi_{acb}$ so that

$$L(e, \phi) = ke\Lambda^{abc}(\phi_{abc} - T_{abc})$$

where Λ^{abc} is defined by

$$\Lambda^{abc} = \frac{1}{4}(\phi^{abc} + \phi^{bac} - \phi^{cab}) + \frac{1}{2}(\eta^{ac}\phi^{b} - \eta^{ab}\phi^{c})$$

and $T_{abc} = e^{\mu}_{b} e^{\nu}_{c} T_{a\mu\nu}$, $\phi_b = \phi^{a}{}_{ab}$.
Variation of the action with respect to ϕ^{abc} yields an equation that can be reduced to $\phi_{abc} = T_{abc}$.

J. W. Maluf and J. F. da Rocha-Neto, Phys. Rev. D 64, 084014 (2001), J. F. da Rocha-Neto, J. W. Maluf, S. C. Ulhoa, Phys. Rev. D 82, 124035 (2010)

Hamiltonian formulation of teleparallel gravity

The momentum canonically conjugated to e_{ak} is

$$\Pi^{ak} = -4ke\Lambda^{a0k}.$$

The first step is obtaining the primary Hamiltonian $H_0 = \Pi^{ai}\dot{e}_{ai} - L$. After, the momentum Π^{ak} is decomposed into irreducible components

$$\Pi^{ak} = e^a_i \Pi^{(ik)} + e^a_i \Pi^{[ik]} + e^a_0 \Pi^{0k}.$$

The definition of this momenta leads to primary constraints $\Gamma^{ab} = 0$

$$\Gamma^{ab} = -\Gamma^{ba} = \Pi^{[ab]} + 4ke(\Sigma^{a0b} - \Sigma^{b0a}),$$

and to $\Pi^{a0} = 0$. Secondary constraints $C^a = 0$ arise from the time evolution of the primary constraints Π^{a0}, that is requiring that $\dot{\Pi}^{a0} = 0$ vanishes. These have the following structure

$$C^a = e^{a0}H_0 + e^{ai}H_i.$$

J. W. Maluf and J. F. da Rocha-Neto, Phys. Rev. D 64, 084014 (2001), J. F. da Rocha-Neto, J. W. Maluf, S. C. Ulhoa, Phys. Rev. D 82, 124035 (2010)

Hamiltonian formulation of teleparallel gravity

From this the full Hamiltonian density, including Lagrange multipliers can be written as

$$H(e_{a\mu}, \Pi^{a\mu}, \lambda_{ab}, \lambda_a) = e_{a0} C^a + \lambda_{ab}\Gamma^{ab} + \lambda_a \Pi^{a0}$$

The first class constraints are searched through the calculation of the Poisson brackets between the constraints,

$$\begin{aligned}
\{C^a(x), C^b(y)\} &= 0, \\
\{C^a(x), \Gamma^{bc}(y)\} &= (\eta^{ab}C^c - \eta^{ac}C^b)\delta(x-y), \\
\{\Gamma^{ab}(x), \Gamma^{cd}(y)\} &= (\eta^{ad}\Gamma^{bc} + \eta^{bc}\Gamma^{ad} - \eta^{ac}\Gamma^{bd} - \eta^{bd}\Gamma^{ac})\delta(x-y).
\end{aligned}$$

All these Poisson brackets are zero when constraints are satisfied. Therefore C^a, Γ^{ab} and π^{a0} are a set of first class constraints, since all Poisson brackets of the constraints Π^{a0} with both C^a and Γ^{ab} vanish strongly.

J. W. Maluf, Annalen Phys. 525 (2013) 339-357

Hamiltonian formulation of teleparallel gravity

The physical degrees of freedom of the theory may be counted in the following way. The pair of dynamical field quantities (e_{ai}, Π^{ai}) displays $12 + 12 = 24$ degrees of freedom. There are $4 + 6 = 10$ first class constraints (C^a, Γ^{ab}) that generate symmetries of the action. Applying the formula,

$$\text{d.o.f.}_{TEGR} = \frac{1}{2}24 - 10 = \mathbf{2}$$

There are two physical degrees of freedom in TEGR, as expected.

J. W. Maluf, Annalen Phys. 525 (2013) 339-357

Hamiltonian formulation of f(T) gravity

Li *et al* (2011) analyze the structure of constraints of $f(\mathcal{T})$ by rewriting its Lagrangian density in an equivalent form

$$L = -e[f(\phi) + (\mathcal{T} - \phi)f'(\phi)]$$

where it can be easily seen that $\phi = T$ if $f''(\phi) = 0$, which reproduces $f(\mathcal{T})$ action.

It can be written as an equivalent Lagrangian density on the form

$$L = -e[\phi\mathcal{T} + V(\phi)]$$

ϕ adds an extra primary constraint $\pi = \dfrac{\partial L}{\partial(\dot{\phi})} = 0$ so now are eleven primary constraints.

However, the appearance of $V(\phi)$ in the Lagrangian and, therefore in the Hamiltonian, modifies the structure of the Poisson brackets.

M. Li, R. X. Miao,Y. G. Miao, JHEP 1107 108 (2011)

Hamiltonian formulation of f(T) gravity

The Poisson brackets of the new primary constraints are long and complicated to calculate. The authors of the work, after some simplifications, find that there are eight second class constraints together with eight first class constraints. Thus using the formula,

$$\text{d.o.f.}_{f(T)} = \frac{1}{2} \times 34 - 8 - \frac{1}{2} \times 8 = \mathbf{5}.$$

they find that $f(\mathcal{T})$ gravity has **3** extra degrees of freedom compared with teleparallel gravity. They do not find a physical interpretation for these extra d.o.f.. Only suggest that they may be related to one massive vector field or one massless vector field plus one scalar field.

M. Li, R. X. Miao, Y. G. Miao, JHEP 1107 108 (2011)

Another form of the lagrangian

Since the tetrad e^a_μ is the primordial field, one may think in writing the TEGR only in terms of the tetrad. We obtain

$$L = eT = e(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu)(\partial_\alpha e^b_\beta - \partial_\beta e^b_\alpha) e^\mu_c e^\nu_e e^\alpha_d e^\beta_f N_{ab}{}^{cdef}$$

where $N_{ab}{}^{cdef}$ is a combination only on Minkowski η, Kronecker delta and Levi-civita symbol in the indices $a, b, c, d, e, f = 0, .., 3$ of the tangent space.

This approach, together with providing a more clean way of looking at the dependence of the lagrangian on the tetrad, it is more suitable for the Hamiltonian procedure.

M. J. Guzmán, R. Ferraro, work in preparation.

Concluding remarks

- Teleparallel gravity and general relativity are related at the level of lagrangians by a surface term.
- Modified teleparallel gravity, or $f(\mathcal{T})$, has second order differential equations of motion, and violated local Lorentz invariance which may be due to its extra degrees of freedom.
- Hamiltonian formulation of teleparallel gravity gives the same two degrees of freedom than general relativity.
- When performing the Hamiltonian decomposition in $f(\mathcal{T})$ gravity, some authors claims that there are three extra degrees of freedom, whose interpretation is still an open question.

Future work

- To find if the extra d.o.f. couples to matter fields.
- Work with the lagrangian form of TEGR only dependent on the tetrad.
- To find if there is a subgroup of the local Lorentz transformations for which the equations of motion of $f(\mathcal{T})$ remains unchanged.
- Kerr-Newmann-like solutions to equations of motion of $f(\mathcal{T})^*$

$*$C. Bejarano, R. Ferraro, M. J. Guzmán, work in preparation. See "Kerr solution in f(T) gravity" at poster session.

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Muito obrigada pela sua atenção!