On the linear stability of the extreme Kerr black hole under axially symmetric perturbations

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Introduction

In all physical theories, studying the solutions of the model is as important as studying its stability. The stability gives a test about the reality of the solutions.
In relativity a lot of work on stability was done, especially in the following cases: (without matter and electromagnetic fields)

- Minkowski
- Schwarzschild
- Kerr

Differences and some results

- For Minkowski there is a global no-linear proof of its stability that was given by D. Christodoulou and S. Klainerman.
- For Schwarzschild the results are not as globals, Bernard S. Kay and Robert M. Wald gave a proof for the stability under linear gravitational perturbations.
- For Kerr, there are some arguments of its stability that rely on the stability of the scalar wave over the Kerr metric. This argument was introduced by Mihalis Dafermos and Igor Rodnianski, but it is not applicable inside the ergo-sphere.

Properties of stationary spaces-times with axial symmetry

For an stationary, axial-symmetric, electro-vacuum space-time, we have only two degrees of freedom

$$\eta \quad \text{and} \quad \omega\,, \tag{1}$$

where η is the square norm of the axial killing vector $\eta^a = \left(\frac{\partial}{\partial\phi}\right)^a$ and ω is its associated scalar twist. For simplicity is convenient write η as

$$\eta = \rho^2 e^\sigma\,. \tag{2}$$

Then all stationary, axial-symmetric, vacuum space-time, can be described by the two functions

$$\sigma \quad \text{and} \quad \omega\,. \tag{3}$$

(2+1)+1 decomposition in the maximal-isothermal gauge

- *(2+1)+1 decomposition*: For a stationary axial-symmetric space-time we can decompose the metric as follows

$$g_{ab} = h_{ab} + \eta_a \eta_b - n_a n_b \tag{4}$$

  Where $h_{\mu\nu}$ is the metric of a two-surface, n_a is a normal vector to the slices at constant t (time coordinate), and η_a is the axial killing vector.

- *Maximal-isothermal gauge*: this gauge consists on the following properties

$$\chi = 0 \quad \text{and} \quad h_{ab} = e^{2u}\delta_{ab}\,. \tag{5}$$

  Where

$$\chi = \chi^{ab} h_{ab}, \quad d\delta^2 = d\rho^2 + dz^2, \quad u = ln\rho + \sigma + q \tag{6}$$

  q is a function that makes u regular at the axis.

The equations

For σ and ω we have the evolution equations

$$- e^{2u}\sigma'' +^{(3)} \Delta\sigma + \partial_A\sigma\frac{\partial^A\bar{\alpha}}{\bar{\alpha}} - 2e^{2u}(\log\rho)'' + 2\frac{\partial_\rho\bar{\alpha}}{\bar{\alpha}\rho} = \left(-e^{2u}(\omega')^2 + |\partial\omega|^2\right)\rho^{-4}e^{-2\sigma} \quad (7)$$

and

$$- e^{2u}\omega'' +^{(3)} \Delta\omega + \partial_A\omega\frac{\partial^A\bar{\alpha}}{\bar{\alpha}} = \frac{2}{\eta}\left(-e^{2u}\omega'\eta' + \partial_A\omega\partial^A\eta\right) \quad (8)$$

where α is the lapse and $\alpha = \rho\bar{\alpha}$. For all scalar function $\psi' = n_\mu\nabla^\mu\psi$ and ${}^{(3)}\Delta = \partial_\rho^2 + \partial_z^2 + \frac{\partial_\rho}{\rho}$

Constraints

The momentum constraint is

$$\partial^B \chi_{AB} = -\frac{e^{2u}}{2\eta^2} \left(\eta' \partial_A \eta + \omega' \partial_A \omega \right) , \tag{9}$$

and the hamiltonian constraint is

$${}^{(3)}\Delta \sigma + \Delta q = -\frac{\epsilon}{4}, \tag{10}$$

where ϵ is the mass density.

The mass

The total ADM mass of the spacetime can be calculated as a volume integral of the mass density ϵ over the half plane $\mathbb{R}^2_+$

$$m = \frac{1}{16}\int_{\mathbb{R}^2_+} \epsilon\, \rho\, d\rho dz \tag{11}$$

where the mass density is

$$\epsilon = \frac{e^{2u}}{\eta^2}\left(\eta'^2 + \omega'^2\right) + |\partial\sigma|^2 + \frac{|\partial\omega|^2}{\eta^2} + 2e^{-2u}\hat{\chi}^{AB}\chi_{AB}. \tag{12}$$

Axial perturbations

We perform the following perturbations for η and ω

$$\sigma = \sigma_0 + \lambda\sigma_1, \tag{13}$$

$$\omega = \omega_0 + \lambda\omega_1 \tag{14}$$

(and similar expressions for the other quantities α, χ, etc.)
The sub-0 stands for for background and sub-1 for perturbations.
In this way the perturbation does not affect the axial symmetry of the background, this means that the perturbation modifies the mass but not the angular momentum.

Extreme Kerr

The evolution equations for σ and ω, are respectively

$$-\frac{e^{2u_0}}{\rho^2}\dot{p} + {}^{(3)}\Delta\sigma_1 = -2\frac{e^{-2\sigma_0}}{\rho^4}(\partial_A\omega_1\partial^A\omega_0 - \sigma_1|\partial\omega_0|^2) \tag{15}$$

$$-\frac{e^{2u_0}}{\rho^2}\dot{d} + {}^{(3)}\Delta\omega_1 = 4\frac{\partial_\rho\omega_1}{\rho} + 2\partial_A\omega_1\partial^A\sigma_0 + 2\partial_A\omega_0\partial^A\sigma_1 \tag{16}$$

with

$$p = \dot{\sigma}_1 - 2\frac{\beta_1^\rho}{\rho} - \beta_1^A D_A\sigma_0 \tag{17}$$

$$d = \dot{\omega}_1 - \beta_1^A D_A\omega_0 \tag{18}$$

where the dot stands for the time derivative.

The Hamiltonian and momentum constraints, are respectively

$$^{(3)}\Delta\sigma_1 + \Delta q_1 = -\frac{1}{2}\left(\partial_A\sigma_0\partial^A\sigma_1 + \left(\partial_A\omega_0\partial^A\omega_1 - \partial_A\omega_0\partial^A\omega_o\right)\rho^{-4}e^{-2\sigma_0}\right) \tag{19}$$

$$\partial^B\chi_{1AB} = -\frac{e^{2u_0}}{2\rho}\left(p\left(2\frac{\partial_A\rho}{\rho} + \partial_A\sigma_0\right) - \frac{\partial_A\omega_0}{\eta_0^2}\tilde{p}\right). \tag{20}$$

The mass density up to second order is

$$\epsilon = \left(\partial\sigma_1 + \omega_1\eta_0\partial\omega_0\right)^2 + \left(\partial\left(\omega_1\eta_0^{-1}\right) - \eta_0^{-1}\sigma_1\partial\omega_0\right)^2 + \left(\eta_0^{-1}\sigma_1\partial\omega_0 - \omega_1\eta_0^{-2}\partial\eta_0\right)^2 \tag{21}$$

where the first order does not contribute to the total mass.

An Interesting Property

Since the background quantities do not depend on time, the form of the equations does not change if we take time derivatives. As an example, if we take the time derivative of the following equation

$$-\frac{e^{2u_0}}{\rho^2}\dot{p} + ^{(3)}\Delta\sigma_1 = -2\frac{e^{-2\sigma_0}}{\rho^4}(\partial_A\omega_1\partial^A\omega_0 - \sigma_1|\partial\omega_0|^2) \tag{22}$$

we get

$$-\frac{e^{2u_0}}{\rho^2}\ddot{p} + ^{(3)}\Delta\dot{\sigma}_1 = -2\frac{e^{-2\sigma_0}}{\rho^4}(\partial_A\dot{\omega}_1\partial^A\omega_0 - \dot{\sigma}_1|\partial\omega_0|^2)\,. \tag{23}$$

As we see, the equations are the same, so we have a new mass density

$$\epsilon_1 = (\partial\dot{\sigma}_1 + \dot{\omega}_1\eta_0\partial\omega_0)^2 + \left(\partial\left(\dot{\omega}_1\eta_0^{-1}\right) - \eta_0^{-1}\dot{\sigma}_1\partial\omega_0\right)^2 + \left(\eta_0^{-1}\dot{\sigma}_1\partial\omega_0 - \dot{\omega}_1\eta_0^{-2}\partial\eta_0\right)^2$$

This means, that we can take time derivatives infinity amount of times and for each resulting set of equations we will get a new mass density.

Main Results

Theorem

Consider a solution σ_1 and ω_1 of the linearized equation. Then, the all the different masses we obtained previously satisfy the following properties.

(i) *They are positive and conserved, and are given by*

$$m = \frac{1}{16}\int_{\mathbb{R}^2_+}\left[\left(\partial\sigma_1 + \omega_1\eta_0\partial\omega_0\right)^2 + \left(\partial\left(\omega_1\eta_0^{-1}\right) - \eta_0^{-1}\sigma_1\partial\omega_0\right)^2 + \left(\eta_0^{-1}\sigma_1\partial\omega_0 - \omega_1\eta_0^{-2}\partial\eta_0\right)^2\right]\rho d\rho dz \quad (24)$$

(ii) *In a domain Ω that does not include the origin, we get the following estimates*

$$|\sigma_1| \leq C_1 m + C_2 m_1, \quad \frac{|\omega_1|}{\rho^4} \leq C_3 m + C_4 m_1 \quad (25)$$

Conclusions

- For the case of Kerr we find a series of positive and conserved quantities that helped to bound the perturbations on a domain that does not include the origin.
- Our method is interesting since it can be applied even within the ergo-sphere.
- At this time we are trying to include the origin in the domain.

Thank you!!