Weyl geometry and scalar-tensor theories

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- Nonmetricity Condition in non-integrable Weyl geometry.

$$\nabla_{\mu} g_{\alpha\beta} = \sigma_{\mu} g_{\alpha\beta} \tag{1}$$

σ as a 1-form field defined on the manifold M.

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- If σ is a *exact form* we have a Integrable Weyl Geometry (IWG).

$$\sigma_\mu = \partial_\mu \phi \tag{4}$$

ϕ as a scalar field.

Integrable Weyl geometry(IWG)

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It is enough to determine a torsionless connection

$$\Gamma^{\alpha}_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\} + \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\mu}\phi_{,\nu} + g_{\beta\nu}\phi_{,\mu} - g_{\mu\nu}\phi_{,\beta} \right), \tag{7}$$

where $\left\{ {\alpha \atop \mu\nu} \right\}$ represents the Chistoffel symbols. Connection coeficients are *invariant* under (5) and (6), as well as *geodesics* are.

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Riemann Frame: Choose $\phi = -f$ implies $\bar{\phi} = 0$

$$\nabla_{\mu}\bar{g}_{\alpha\beta} = 0 \tag{8}$$

where $\bar{g}_{\alpha\beta} = e^{-\phi} g_{\alpha\beta}$. This effective metric will help us to construct invariant quantities in the Weyl sense.

In this framework it is reasonable to change the definitions of physical quantities (to maintain invariance)

- Curvature tensors: $R^{\alpha}{}_{\beta\mu\nu}$, $R_{\beta\nu} = R^{\alpha}{}_{\beta\alpha\nu}$ and $e^{\phi}R$ are weyl-invariant.
- We redefine the arc length (propertime) of a curve $\tau = \int e^{-\frac{\phi}{2}} \sqrt{g_{\mu\nu} \frac{dx^{u}}{d\lambda} \frac{dx^{\nu}}{d\lambda}} d\lambda.$
- We redefine the p-dimensional volume form to $\Omega = \sqrt{-g} e^{-\frac{p}{2}\phi} dx^{1} \wedge ... \wedge dx^{p}$

Characteristics of Brans-Dicke gravity

- A metric g and a scalar field Φ, being the connection the Levi-Civita one, $\nabla = {}_g\nabla$

$$S = \int d^4x \frac{\sqrt{-g}}{16\pi}(\Phi R + \frac{\omega}{\Phi}\Phi^{,\alpha}\Phi_{,\alpha} + 16\pi L_m). \quad (9)$$

  being the Brans-Dicke scalar field such that $\Phi = \frac{1}{G}$.

- The theory allows for transformations (rescaling) of g and Φ,

$$\bar{g}_{\mu\nu} = \frac{\Phi}{\Phi_0} g_{\mu\nu} \qquad \varphi = \sqrt{\frac{2\omega + 3}{16\pi G}} \ln \frac{\Phi}{\Phi_0} \quad (10)$$

- Einstein frame formulation (More than GR with a massless SF)

$$S = \int d^4x\sqrt{-\bar{g}}\left(\frac{\bar{R}}{16\pi G} + \frac{1}{2}\varphi_\mu\varphi^\mu\right) + \int d^4x\sqrt{-\bar{g}}e^{-8\sqrt{\frac{\pi G}{2\omega - 3}}\varphi}L_m (11)$$

Conformal frames

- In Jordan frame the affine connection is Levi-Civita of g, so Weyl field is $\phi = 0$, ($\nabla_\alpha g_{\mu\nu} = 0$), this corresponds to what we have called the *Riemann frame* in a IWG. With minimal coupling to matter (Satisfies WEP)
- In Einstein frame the Levi-Civita connection of $\bar{g}$ has addictional terms arising from Rieman frame to *Weyl frame*,($\nabla_\alpha \bar{g}_{\mu\nu} = \phi_\alpha g_{\mu\nu}$). Sometimes interpreted as a (fifth) force (Do no satisfies WEP).
- There is the, sometimes confuse, debate about what of these frame is the physical one. To Dicke (1962) *"It is evident that the particular values of the units of mass, length and time employed are arbitrary and that the laws of physics must be invariant under a general coordinate dependent change of units"*.

BD in Einstein's frame is similar to WIST

- In investigating Weyl Integrable spacetimes (WIST), Novello et. al. have found the Weyl field is able to avoid initial cosmic singularity (big bang),a exemple of bouncing universes. The action of WIST is (Salim-Sautú) [?]

$$S = \int d^4x\sqrt{-g}(R + \xi\nabla_\alpha\phi^{,\alpha} + e^{-2\phi}L_m). \tag{12}$$

  where ∇ is the weylian conection, $\nabla_\alpha\phi^{,\alpha} = \phi^{,\alpha};_\alpha - 2\phi^\alpha\phi_\alpha$. In terms of riemannian curvature, $R = \bar{R} - 3\Box\phi + \frac{3}{2}\phi_\alpha\phi^\alpha$

- Matter couples directly to scalar field. In a Weyl approach we have a prescription to fix the motion on Weyl geodesics. $\eta \to e^{-\phi}g$ and $\partial \to \nabla_{Weyl}$.

Palatini variational method in BD Action

- Making use of the field transformation $\Phi = e^{-\phi}$, (**??**) can be rewritten as

$$S_G = \int d^4x\sqrt{-g}e^{-\phi}(R + \omega\phi^{,\alpha}\phi_{,\alpha}). \tag{13}$$

- Now we can derive in a dynamical way the Integrable Weyl Geometry (IWG) using Palatini method (extended)(Weyl frame)

$$\nabla_\alpha(\sqrt{-g}e^{-\phi}g^{\mu\nu}) = 0. \tag{14}$$

- We have to change matter coupling if we intent satisfy the WEP: the hint comes from the effective metric $\sqrt{-g}e^{-\phi}g^{\mu\nu} = \sqrt{-\gamma}\gamma^{\mu\nu}$

$$S_m = \kappa \int d^4x\sqrt{-\gamma}L_m(\gamma_{\mu\nu}, \Psi, \nabla^{(\gamma)}\Psi) \tag{15}$$

$$= \kappa \int d^4x\sqrt{-g}e^{-2\phi}L_m(e^{-\phi}g_{\mu\nu}, \Psi, \nabla^{(e^{-\phi}g)}\Psi),$$

Field equations

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= -\kappa T^{(\gamma)}_{\mu\nu} - \omega(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi^{,\alpha}\phi_{,\alpha}), \\
\Box\phi + \phi^{,\alpha}\phi_{,\alpha} &= 0,
\end{aligned} \tag{16}
$$

Riemannian expression with the BD variable $\Phi = e{-}\phi$

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu} &- \frac{2\omega - 3}{2\Phi^2}\left(\Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\Phi^{,\alpha}\Phi_{,\alpha}\right) \\
&- \frac{1}{\Phi}\left(\nabla_\mu\Phi_\nu - g_{\mu\nu}\Box\Phi\right) \\
&\Box\Phi = 0,
\end{aligned} \tag{17}
$$

With energy momentum-tensor constructed by Weyl prescription.

In going to Riemann frame we have *RG + massless scalar field*

- Fisher-Janis-Newman-Winnicor-Wyman solution: compatible with solar system tests.

$$
\begin{aligned}
d\bar{s}^2 &= W^S dt^2 - W^{-S} dr^2 - r^2 W^{1-S}(d\theta^2 + \sin^2\theta d\varphi^2) \\
\phi(r) &= -\frac{1}{2\eta} \ln |W|
\end{aligned}
$$

$W = 1 - \frac{r_0}{r}$ with $r_0 = 2\eta$ and $\eta = \sqrt{M^2 + \omega}$ also
$S = \frac{M}{\eta} = \frac{M}{\sqrt{M^2 + \omega/2}}$

- Naked singularities: Violating the Cosmic censorchip Conjecture in a different (geometrical) way. Wyman wormholes (Not traversable by humans) (TSA and Formiga, to be published).

- We use Weyl invariant quantities to characterize physical

- Completly geometrical description of gravity
- Particle's motion is governed by weyl geodesics.
- To couple matter and Weyl scalar field
- Another view on astrophysical objects
- Explore the redefinitions of physical quantities as area of event horizon, entropy,etc

- Thank you!